

When does the Price of Anarchy tend to 1 in Large Walrasian Auctions and Fisher Markets?*

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Abstract

As is well known, many classes of markets have efficient equilibria, but this depends on agents being non-strategic, i.e. that they declare their true demands when offered goods at particular prices, or in other words, that they are price-takers. An important question is how much the equilibria degrade in the face of strategic behavior, i.e. what is the Price of Anarchy (PoA) of the market viewed as a mechanism?

Often, PoA bounds are modest constants such as $\frac{4}{3}$ or 2. Nonetheless, in practice a guarantee that no more than 25% or 50% of the economic value is lost may be unappealing. This paper asks whether significantly better bounds are possible under plausible assumptions. In particular, we look at how these worst case guarantees improve in the following large settings.

- Large Walrasian auctions: These are auctions with many copies of each item and many agents. We show that the PoA tends to 1 as the market size increases, under suitable conditions, mainly that there is some uncertainty about the numbers of copies of each good and demands obey the gross substitutes condition. We also note that some such assumption is unavoidable.
- Large Fisher markets: Fisher markets are a class of economies that has received considerable attention in the computer science literature. A large market is one in which at equilibrium, each buyer makes only a small fraction of the total purchases; the smaller the fraction, the larger the market. Here the main condition is that demands are based on homogeneous monotone utility functions that satisfy the gross substitutes condition. Again, the PoA tends to 1 as the market size increases.

Furthermore, in each setting, we quantify the tradeoff between market size and the PoA.

1 Introduction

When is there no gain to participants in a game from strategizing? One answer applies when players in a game have no prior knowledge; then a game that is strategy proof ensures that truthful actions are a best choice for each player. However, in many settings there is no strategy proof mechanism. Also, even if there is a strategy proof mechanism, with knowledge in hand, other equilibria are possible, for example, the “bullying” Nash Equilibrium as illustrated by the following example: there is one item for sale using a second price auction, the low-value bidder bids an amount at

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least equal to the value of the high-value bidder, who bids zero; the resulting equilibrium achieves arbitrarily small social welfare compared to the optimal outcome.

To make the notion of gain meaningful one needs to specify what the game or mechanism is seeking to optimize. Social welfare and revenue are common targets. For the above example, the social welfare achieved in the bullying equilibria can be arbitrarily far from the optimum. However, for many classes of games, over the past fifteen years, bounds on the gains from strategizing, a.k.a. the Price of Anarchy (PoA), have been obtained, with much progress coming thanks to the invention of the smoothness methodology [1, 2, 3, 4]; many of the resulting bounds have been shown to be tight. Often these bounds are modest constants, such as $\frac{4}{3}$ [5] or 2 [6], etc., but rarely are there provably no losses from strategizing, i.e. a PoA of 1.

This paper investigates when bounds close to 1 might be possible. In particular, we study both large Walrasian auctions and large Fisher markets viewed as mechanisms.

Walrasian Auctions They are used in settings where there are goods for sale and agents, called bidders, who want to buy these goods. Each agent has varying preferences for different subsets of the goods, preferences that are represented by valuation functions. The goal of the auction is to identify equilibrium prices; these are prices at which all the goods sell, and each bidder receives a favorite (utility maximizing) collection of goods, where each bidder's utility is quasi-linear: the difference of its valuation for the goods and their cost at the given prices. Such prices, along with an associated allocation of goods, are said to form a Walrasian equilibrium.

Walrasian equilibria for indivisible goods are known to exist when each bidder's demand satisfies the gross substitutes property [7], but this is the only substantial class of settings in which they are known to exist.

[8] analyzed the PoA of the games induced by Walrasian mechanisms, i.e. the prices were computed by a method, such as an English or Dutch auction, that yields equilibrium prices when these exist. Note that the mechanism can be applied even when Walrasian equilibria do not exist, though the resulting outcome will not be a Walrasian equilibrium. But even when Walrasian equilibria exist, because bidders may strategize, in general the outcome will be a Nash equilibrium rather than a Walrasian one. Among other results, Babaioff et al. showed an upper bound of 4 on the PoA for any Walrasian mechanism when the bids and valuations satisfied the gross substitutes property and overbidding was not allowed.¹ In addition, they obtained lower bounds on the PoA that were greater than 1, even when overbidding was not allowed, which excludes bullying equilibria; e.g. the English auction has a PoA of at least 2.

Babaioff et al. also noted that the prices computed by double auctions, widely used in financial settings, are essentially computing a price that clears the market and maximizes trade; one example they mention is the computation of the opening prices on the New York Stock Exchange, and another is the adjustment of prices of copper and gold in the London market.

By a large auction, we intend an auction in which there are many copies of each good, and in addition the demand set of each bidder is small. The intuition is that then each bidder will have a small influence and hence strategic behavior will have only a small effect on outcomes. In fact, this need not be so. For example, the bullying equilibrium persists: it suffices to increase the numbers of items and bidders for each type to n , and have the buyers of each type follow the same strategy as before.

What allows this bullying behavior to be effective is the precise match between the number of items and the number of low-value bidders. The need for this exact match also arises in the lower-bound examples in [8] (as with the bullying equilibrium, it suffices to pump up the examples by a

¹They also proved a version of this bound which was parameterized w.r.t. the amount of overbidding.

factor of n). To remove these equilibria that demonstrate PoA values larger than 1, it suffices to introduce some uncertainty regarding the numbers of items and/or bidders. Indeed, in a large setting it would seem unlikely that such numbers would be known precisely. We will create this uncertainty by using distributions to determine the number of copies of each good. This technique originates with [9]. In contrast, prior work on non-large markets eliminated the potentially unbounded PoA of the bullying equilibrium by assuming bounds on the possible overbidding [10, 11, 12, 3].

Our main result on large Walrasian auctions is that the PoA of the Walrasian mechanism tends to 1 as the market size grows. This result assumes that expected valuations are bounded regardless of the size of the market. We specify this more precisely when we state our results in Section 3. This bound applies to both Nash and Bayes-Nash equilibria; as it is proved by means of a smoothness argument, it extends to mixed Nash and coarse correlated equilibria, and outcomes of no-regret learning.

Fisher Markets A Fisher market is a special case of an exchange economy in which the agents are either buyers or sellers. Each buyer is endowed with money but has utility only for non-money goods; each seller is endowed with non-money goods, WLOG with a single distinct good, and has utility only for money. Fisher markets capture settings in which buyers want to spend all their money. In particular, they generalize the competitive equilibrium from equal incomes (CEEI) [13, 14], in that they allow buyers to have non-equal incomes. While at first sight this might appear rather limiting, we note that much real-world budgeting in large organizations treats budgets as money to be spent in full, with the consequence that unspent money often has no utility to those making the spending decisions. The budgets in GoogleAds and other online platforms can also be viewed as money that is intended to be spent in full.

We consider the outcomes when buyers bid strategically in terms of how they declare their utility functions. We show that the PoA tends to 1 as the setting size increases. The only assumptions we need are some limitations on the buyers' utility functions: they need to satisfy the gross substitutes property and to be monotone and homogeneous of degree 1.

This result is also obtained via a smoothness-type bound and hence extends to bidders playing no-regret strategies, assuming that the ensuing prices are always bounded away from zero. We ensure this by imposing reserve prices, and this result is deferred to the appendix.

Roadmap In Section 2 we provide the necessary definitions and background, in Section 3 we state our results, which are then shown in Sections 4 and 5, covering large auctions and large Fisher markets respectively.

1.1 Related Work

The results on auctions generalize earlier work of [9] who showed analogous results for auctions of multiple copies of a single good. In contrast, we consider auctions in which there are multiple goods. Swinkels analyzed discriminatory and non-discriminatory mechanisms. For the latter, he showed that any mechanism that used a combination of the k -th and $(k+1)$ -st prices when there were k copies of the good on sale achieved a PoA that tended to 1 with the auction size². Our result also weakens some of the assumptions in Swinkels work.

The second closely related work on auctions is due to [4]. They also consider large settings and show that for several market settings when using simple, non-Walrasian mechanisms, the PoA tends to 1 as the market size grows to ∞ . Their results are derived from a new type of smoothness

²Swinkels did not use the then recently formulated PoA terminology to state his result.

argument. Depending on the result, they require either uncertainty in the number of goods or the number of bidders. In contrast, our main result uses a previously known smoothness technique. They also show that for traffic routing problems, the PoA of the atomic case tends to that of the non-atomic case as the number of units of traffic grows to ∞ .

The idea of uncertainty in the number of agents or items first arose in the Economics literature. [15] used it in the context of voting games, and [9] in the context of auctions. Later, uncertainty in the number of agents was used with the Strategy Proof in the Large concept [16].

The study of the behavior of large exchange economies was first considered by [17], which they modeled as a *replica economy*, the n -fold duplication of a base economy, showing that individual utility gains from strategizing tend to zero as the economy grows. Subsequently, [18] showed that with some *regularity* assumptions, the equilibrium allocations converge to the competitive equilibrium. More recently, [19] studied the efficiency of exchange economies in the presence of strategic agents; however, their notion of efficiency was weaker than the PoA. They termed an outcome μ -efficient if there was no way of improving everyone's outcome in terms of utility by an additive factor of μ , and showed that with high probability (i.e. $1 - \mu$) a μ -efficient outcome would occur when the size of the economy was large enough, so long as each agent was small, agents were truthful with non-zero probability, and some additional more technical conditions. In contrast, the PoA considers the ratio of the optimal social welfare to the achieved social welfare, namely a ratio of the sum of everyone's outcomes.

[20] analyzed the PoA of strategizing in Fisher markets. The PoA compared the social welfare of the worst resulting Nash Equilibrium to the optimal, i.e. welfare maximizing assignment, under a suitable normalization of utilities. Among other results, they showed lower bounds of $\Omega(\sqrt{n})$ on the PoA when there are n buyers with linear utilities. However, we view the comparison point of an optimal assignment to be too demanding in this setting, as it may not be an assignment that could arise based on a pricing of the goods. In our results we will be comparing the strategic outcomes to those that occur under truthful bidding. Another approach is to bound the gains to individual agents, called the *incentive ratio*; [21, 22] showed these values were bounded by small constants in Fisher market settings.

There has been much other work on large settings and their behavior. We mention only a sampling. [23] studied the notion of extensive robustness for large games, and [24] investigated large repeated games using the notion of compressed equilibria. [25] studied repeated games and the use of differential privacy as a measure of largeness. In a different direction, [26] investigated fault tolerance in large games for λ -continuous and anonymous games.

2 Preliminaries

2.1 Definitions for Large Walrasian Auctions

Definition 2.1. *An auction A comprises a set of N bidders B_1, B_2, \dots, B_N , and a set of m goods G , with n_j copies of good j , for $1 \leq j \leq m$. We write $\mathbf{n} = (n_1, n_2, \dots, n_m)$, where n_j denotes the number of copies of good j , and we call it the multiplicity vector. We also write $\mathbf{n} = (n_j, n_{-j})$, where n_{-j} is the vector denoting the number of copies of goods other than good j . We refer to an instance of a good as an item. For an allocation x_i to bidder i , which is a subset of the available goods, we write $\mathbf{x}_i = (x_{i1}, x_{i2}, \dots, x_{im})$ where x_{ij} denotes the number of copies of good j in allocation \mathbf{x}_i . There is a set of prices $\mathbf{p} = (p_1, p_2, \dots, p_m)$, one per good; we also write $\mathbf{p} = (p_j, p_{-j})$. Each bidder i has a valuation function $v_i : X \rightarrow \mathbb{R}_+$, where X is the set of possible assignments, and a quasi-linear utility function $u_i(\mathbf{x}_i) = v_i(\mathbf{x}_i) - \mathbf{x}_i \cdot \mathbf{p}$.*

A Walrasian equilibrium is a collection of prices \mathbf{p} and an allocation \mathbf{x}_i to each bidder i such that

(i) the goods are fully allocated but not over-allocated, i.e. for all j , $\sum_i x_{ij} \leq n_j$, and $\sum_i x_{ij} = n_j$ if $p_j > 0$, and (ii) each bidder receives a utility maximizing allocation at prices \mathbf{p} , i.e. $u_i(\mathbf{x}_i) = v_i(\mathbf{x}_i) - \mathbf{x}_i \cdot \mathbf{p} = \max_{\mathbf{x}_i} [v_i(\mathbf{x}_i) - \mathbf{x}_i \cdot \mathbf{p}]$.

In a Walrasian mechanism for auction A each bidder produces a bid function $b_i : X \rightarrow \mathbb{R}_+$. We write $\mathbf{b} = (b_1, b_2, \dots, b_N)$ and $\mathbf{b}_i = (b_i, b_{-i})$. The mechanism computes prices and allocations as if the bids were the valuations.

Given the bidders and their bids, $\mathbf{p}(\mathbf{n}; \mathbf{b})$ denotes the prices produced by the Walrasian mechanism at hand when there are \mathbf{n} copies of the goods and \mathbf{b} is the bidding profile. Also, $p_j(\mathbf{n}; \mathbf{b})$ denotes the price of good j and $\mathbf{p}(\mathbf{n}; \mathbf{b}) = (p_j(\mathbf{n}; \mathbf{b}), p_{-j}(\mathbf{n}; \mathbf{b}))$. Finally, we let both $\mathbf{x}_i(\mathbf{n}; \mathbf{b})$ and $\mathbf{x}_i(\mathbf{n}; b_i, b_{-i})$ denote the allocation to bidder i provided by the mechanism.

Definition 2.2. A valuation or bid function satisfies the gross substitutes property if for each utility maximizing allocation \mathbf{x} at prices $\mathbf{p} = (p_j, p_{-j})$, at prices (q_j, p_{-j}) such that $q_j > p_j$, there is a utility maximizing allocation \mathbf{y} with $y_{-j} \succeq x_{-j}$ (i.e. $y_k \geq x_k$ for $k \neq j$).

Definition 2.2 applies to the Fisher market setting also.

A large Walrasian auction is an auction with N bidders where N is large. However, in order to state theorems parameterized by N , we define a large auction as being a sequence of auctions with increasing numbers of bidders, as follows.

Definition 2.3. A large Walrasian auction is a sequence of auctions $A_1, A_2, \dots, A_N, \dots$, where N denotes the number of bidders. It satisfies the following two properties.

- i. The demand of every bidder is for at most k items. Formally, if allocated a set of more than k items, the bidder will obtain equal utility with a subset of size k .
- ii. Let $F(n_j, j, N | n_{-j})$ denote the probability that there are exactly n_j copies of good j when given n_{-j} copies of other goods, and let $F(N) = \max_j \max_{n_j, n_{-j}} F(n_j, j, N | n_{-j})$. Then, for all n , $\lim_{N \rightarrow \infty} F(N) = 0$.

A Bayes-Nash equilibrium is an outcome with no expected gain from an individual deviation:

$$\forall b'_i : \mathbb{E}_{\mathbf{n}, v_{-i}, b_{-i}} [u_i(x_i(\vec{n}; b_i, b_{-i}), p((b_i, b_{-i})))] \geq \mathbb{E}_{\mathbf{n}, v_{-i}, b_{-i}} [u_i(x_i(\vec{n}; b'_i, b_{-i}), p((b'_i, b_{-i})))] .$$

The social welfare $SW(\mathbf{x})$ of an allocation \mathbf{x} is the sum of the individual valuations: $SW(\mathbf{x}) = \sum_i v_i(\mathbf{x}_i)$. We also write $SW(OPT)$ for the (expected) optimal social welfare, the maximum (expected) achievable social welfare, and $SW(NE)$ for the smallest (expected) social welfare achievable at a Bayes-Nash equilibrium.

Finally, the Price of Anarchy is the worst case ratio of $SW(OPT)$ to $SW(NE)$ over all instances in the class of games at hand, which in this context comprise auctions A_N of N buyers:

$$PoA = \max_{A_N} \frac{SW(OPT)}{SW(NE)} .$$

2.2 Definitions for Large Fisher Markets

Definition 2.4. A Fisher market³ has m divisible goods and N agents, called buyers. There is a fixed endogenous supply of each good (which WLOG is chosen to be 1 unit). Agent i has a

³In much of the Computer Science literature the term market has been used to mean what is called an economy in the economics literature.

fixed endowment of e_i units of money. Each agent has a utility function, with the characteristic that the agent has no desire for its money, i.e. each agent seeks to spend all its money on goods. Suppose we assign a price p_j to each good j , then a (possibly non-unique) demand of agent i is a bundle of goods $(x_{i1}, x_{i2}, \dots, x_{im})$ that maximizes her utility subject to the budget constraint: $\sum_j p_j x_{ij} \leq e_i$. A market demand $x_{\{j\}}$ for a good j is the total (possibly non-unique) demand for that good; $x_{\{j\}} = \sum_i x_{ij}$. This is viewed as a function of the price vector $\mathbf{p} = (p_1, p_2, \dots, p_m)$. Prices \mathbf{p} provide an equilibrium if the resulting markets can clear, that is there exists a market demand at these prices such that for all j , $x_{\{j\}} = 1$ if $p_j > 0$. For notational convenience, we define an excess demand for good j as $z_{\{j\}} = x_{\{j\}} - 1$. The equilibrium condition is that every excess demand be zero. Prices \mathbf{p} form an equilibrium if there exists market demands at these prices that clear the market.

The Fisher market is actually a special case of an Exchange economy. (To see this, view the money as another good, and the supply of the goods as being initially owned by another agent, who desires only money.)

In general computing equilibria is computationally hard even for Fisher markets [27, 28]. One feasible class is the class of Eisenberg-Gale markets, markets for which the equilibrium computation becomes the solution to a convex program. This class was named in [29]; the program was previously identified in [30].

Definition 2.5. Eisenberg-Gale markets are those economies for which the equilibria are exactly the solutions to the following convex program, called the Eisenberg-Gale convex program:

$$\begin{aligned} \max_{\mathbf{x}} \quad & \sum_{i=1}^n e_i \cdot \log(u_i(x_{i1}, x_{i2}, \dots, x_{im})) \\ \text{s.t.} \quad & \forall j : \quad \sum_i x_{ij} \leq 1 \\ & \forall i, j : \quad x_{ij} \geq 0. \end{aligned} \tag{2.1}$$

In a Fisher market game, each buyer produces a bid function b_i . The mechanism computes prices and allocations as if the bids were the valuations. The same restrictions will apply to the bid functions and the utilities.

Notational remark The demands are induced by the bids, thus we could write $u_i(\mathbf{x}_i(b_i, b_{-i}))$, but for brevity we will write this as $u_i(b_i, b_{-i})$ instead. Also, it will be convenient to write v_i for the truthful bid of u_i , yielding the notation $u_i(v_i, v_{-i})$.

Definition 2.6. The size L of a Fisher market is defined to be the ratio $L = \frac{\sum_i e_i}{\max_i e_i}$.

It is natural to measure the efficiency of outcomes in the Fisher market game using the objective function (2.1), or rather its exponentiated form. More specifically, we compare the geometric means of the buyer's utilities weighted by their budgets at the worst Nash Equilibrium (with bids b) and at the market equilibrium (with bids v).

$$\min_{\text{NE with bids } b} \left(\prod_i \left(\frac{u_i(b_i, b_{-i})}{u_i(v_i, v_{-i})} \right)^{e_i} \right)^{\frac{1}{\sum_i e_i}}.$$

Note that in the settings we consider the prices at a market equilibrium are unique. We will also use this product to upper bound a Price of Anarchy notion for a market M , which compares

the sum of the utilities at the worst Nash Equilibrium to the sum at the market equilibrium.

$$PoA(M) = \min_{NE \text{ with bids } \mathbf{b}} \frac{\sum_i u_i(b_i, b_{-i})}{\sum_i u_i(v_i, v_{-i})}.$$

For the latter measure to be meaningful, we need to use a common scale for the different buyers' utilities. To this end, we define consistent scaling.

Definition 2.7. *The bidders' utilities are consistently scaled if there is a parameter $t > 0$ such that for every bidder i , $u_i(v_i, v_{-i}) = t e_i$.⁴ That is, bidder i 's utility function is scaled to give it utility $t e_i$ at the market equilibrium, where e_i is its budget.*

Finally, we will be considering utility functions that are monotone, homogeneous of degree one, as defined below, continuous, concave, and that induce demands that satisfy the gross substitutes condition (see Definition 2.2).

Definition 2.8. *Utility function $u(\mathbf{x})$ is homogeneous of degree 1 if for every $\alpha > 0$, $u(\alpha \mathbf{x}) = \alpha \cdot u(\mathbf{x})$.*

Fact 2.1. *The utility functions in Eisenberg Gale programs are assumed to be homogeneous of degree 1, continuous and concave.*

2.3 Regret Minimization

In a regret minimization setting, a single player is playing a repeated game. At each round, the player can choose to play one of K strategies, which are the same from round to round. The outcome of the round is a payoff in the range $[-\chi, \chi]$.

Definition 2.9. *An algorithm that chooses the strategy to play is regret minimizing if the outcome of the algorithm, in expectation, is almost as good as the outcome from always playing a single strategy regardless of any one else's actions; namely, for any b_{-i}^t , for any fixed strategy $b_i \in K$,*

$$\sum_{t=1}^T u_i(b_i^t, b_{-i}^t) - \sum_{t=1}^T u_i(b_i, b_{-i}^t) \geq -\Phi(|K|, T) \cdot \chi,$$

where $\Phi(|K|, T) = o(T)$ and b_i^t is the strategy bidder i uses at time t .

Theorem 2.1. *Regret minimizing algorithms exist. If, at the end of each round, the player learns the payoff for all K strategies, $\Phi(|K|, T) = O(\sqrt{T})$ can be achieved, and if she learns just the payoff for her strategy, $\Phi(|K|, T) = O(T^{\frac{2}{3}})$ can be achieved.*

Note that in large auctions and markets, it is the latter result that seems more applicable.

As shown in [1], if all players play regret minimizing strategies, the resulting outcome observes the PoA bound obtained via a smoothnes argument up to the regret minimization error.

⁴WLOG, we may assume that $t = 1$.

3 Our Results

One issue that deserves some consideration when specifying a large setting, and placing some inevitable restrictions on the possible settings, is to determine which parameters should remain bounded as the setting size grows. So as to be able to state asymptotic results, we give results in terms of a parameter N which is allowed to grow arbitrarily large. But in fact all settings are finite, so really when stating that some parameters are bounded, we are making statements about the relative sizes of different parameters.

One common assumption is that the type space is finite. However, it is not clear such an assumption is desirable in the settings we consider, for it would be asserting that the number of possible valuations and bidding strategies is much smaller than the number of bidders. Another standard assumption is that the ratio of the largest to smallest non-zero valuations are bounded. This, for example, would preclude valuations being distributed according to a power law distribution (or any other unbounded distribution), which again seems unduly restrictive if it can be avoided.

3.1 Result for Walrasian Auctions

Our analysis makes two assumptions; stronger assumptions were made for the large auction results in [9, 4]. [9] also ruled out overbidding by arguing it is a dominated strategy. Our analysis can avoid even this assumption of other players' rationality, however, bounded overbidding is needed for the extension to regret minimizing strategies.

Assumption 3.1. [Bounded Expected Valuation] *There is a constant ζ such that for each bidder and each item, her expected value for this single item is at most ζ :*

$$\max_s \mathbb{E}_{v_i}[v_i(s)] \leq \zeta.$$

Note that without this assumption the social welfare would not be bounded, and then it is not clear how to measure the Price of Anarchy. Prior work had assumed $v_i(s) \leq \zeta$ for all s and i (i.e. absolutely rather than in expectation).

Assumption 3.2. [Market Welfare] *The optimal social welfare grows linearly with the number of bidders: $\text{SW}(\text{OPT}) \geq \rho N$, for some constant $\rho > 0$.*

[4] also makes this assumption. [9] makes assumptions on the value distribution which imply Assumption 3.2 although this consequence is not stated in his work.

We can achieve Assumption 3.2 by making following assumptions.

Assumption 3.3. [Auction Size] *Let $\mu(n_j)$ be the expected number of copies of good j , for $1 \leq j \leq l$, and let $\Gamma(n_j)$ be its standard deviation. The assumption is that for each j , $\mu(n_j) = \Theta(N)$ and $\Gamma(n_j) \leq (1 - \lambda)\mu(n_j)$ for some constant $\lambda > 0$. Let $\alpha > 0$ be such that $\mu(n_j) \geq \alpha N$ for all j and sufficiently large N .*

Assumption 3.4. [Value Lower Bound] *There is a parameter $\rho' > 0$ such that for any bidder, its largest expected value for one item is at least ρ' :*

$$\max_s \mathbb{E}_{v_i}[v_i(s)] \geq \rho'.$$

Lemma 3.1. *Let $\rho = \lambda^2 \alpha \frac{2\lambda + \lambda^2}{(1 + \lambda)^2} \rho'$. If Assumptions 3.3 and 3.4 hold, then $\text{SW}(\text{OPT}) \geq \rho N$.*

Theorem 3.1. *In a large Walrasian auction which satisfies Assumptions 3.1 and 3.2 and with buyers whose valuation and bid functions are monotone and satisfy the gross substitutes property,*

$$\text{SW(NE)} \geq \left(1 - \frac{3 \cdot k \cdot (k+1) \cdot \zeta \cdot m}{\rho} \cdot Y \cdot \lceil \log_2 \frac{1}{Y} \rceil\right) \text{SW(OPT)},$$

where $Y = m \cdot F(N) \left[2m \binom{k+1+m}{m} \right]$.

In particular, if the number of copies of each good is independently and identically distributed according to the Binomial distribution $B(N, \frac{1}{2})$, and $k, m = O(1)$, then

$$\text{SW(NE)} \geq \left(1 - O\left(\frac{\log N}{\sqrt{N}}\right)\right) \text{SW(OPT)}.$$

Also, if there is only one good, i.e., if $m = 1$, then

$$\text{SW(NE)} \geq \left(1 - \frac{3 \cdot k \cdot (k+1) \cdot \zeta}{\rho} \cdot Y \cdot \lceil \log_2 \frac{1}{Y} \rceil\right) \text{SW(OPT)},$$

where $Y = 2(k+2) \cdot F(N)$.

Definition 3.1. *Let K be the set of strategies a player uses when running a regret minimization algorithm. She is a (γ, δ) -player if $v \in K$ and, for any $b \in K$ and for any set \mathbf{x} ,*

$$b(\mathbf{x}) \leq v(\mathbf{x}) \cdot \gamma + \delta.$$

Theorem 3.2. *Suppose all players use regret minimization algorithms, they are all (γ, δ) -players and their valuation and bid functions are monotone and satisfy the gross substitutes property. Then, in a large Walrasian auction which satisfies Assumptions 3.1 and 3.2,*

$$\frac{1}{T} \mathbb{E}_{\mathbf{n}, \mathbf{v}, \mathbf{b}} \left[\sum_{t=1}^T v_i(\mathbf{x}_i(b_i^t, b_{-i}^t)) \right] \geq \left(1 - \frac{3 \cdot k \cdot (k+1) \cdot \zeta \cdot m}{\rho} \cdot Y \cdot \lceil \log_2 \frac{1}{Y} \rceil - \frac{\max_i \Phi(|K_i|, T) \cdot (km\zeta\gamma + \delta)}{\rho \cdot T}\right) \text{SW(OPT)}.$$

where K_i is the set of strategies used by i and $v_i \in K_i$.

3.2 Fisher Market Results

Theorem 3.3. *Let M be a large Fisher market with largeness L in which the utility functions and bid functions are homogeneous of degree 1, concave, continuous, monotone and satisfy the gross substitutes property. If its demands as a function of the prices are unique at any $\mathbf{p} > 0$, or if its utility functions are linear, then its Price of Anarchy is bounded by*

$$\text{PoA}(M) \leq e^{m/L},$$

where m is the number of distinct goods in the market.

Perhaps surprisingly, there is no need for uncertainty in this setting. Note that these assumptions on the utility functions are satisfied by Cobb-Douglas utilities, and by those CES and Nested CES utilities that meet the weak gross substitutes condition.

Theorem 3.4. Suppose all players use regret minimization algorithms, their utility functions and bid functions are homogeneous of degree 1, concave, continuous, monotone, and satisfy the gross substitutes property. If its demands as a function of the prices are unique at any $\mathbf{p} > 0$, or if its utility functions are linear, then in a large Fisher Market with largeness L with reserve prices \mathbf{r} such that for any j , $\frac{1}{\lambda} \leq \frac{r_j}{p_j(\mathbf{v})} \leq \frac{1}{4}$,

$$\frac{1}{T} \sum_{t=1}^T \sum_i u_i(b_i^t, b_{-i}^t) \geq (e^{-\frac{2m}{L}} - \frac{\max_i \Phi(|K_i|, T)}{T} \lambda) \sum_i u_i(v_i, v_{-i}),$$

where K_i is the set of strategies used by player i and $v_i \in K_i$.

4 Walrasian Equilibria

Recall that the English Walrasian mechanism can be implemented as an ascending auction. The prices it yields can be computed as follows: p_j is the maximum possible increase in the social welfare when the supply of good j is increased by one unit. Similarly, the Dutch Walrasian mechanism can be implemented as a descending auction, and the resulting price p_j is the loss in social welfare when the supply of good j is decreased by one unit.

We will be considering an arbitrary Walrasian mechanism. Necessarily, its prices must lie between those of the Dutch Walrasian and English Walrasian mechanisms. We let $\mathbf{p}^{Eng}(\mathbf{n}; (b_i, b_{-i}))$ denote the price output by the English Walrasian mechanism and $\mathbf{p}^{Dut}(\mathbf{n}; (b_i, b_{-i}))$ be the price output by the Dutch Walrasian mechanism.

We define the distance between two price vectors \mathbf{p} and \mathbf{p}' with respect to U as follows:

$$dist^U(\mathbf{p}, \mathbf{p}') = \sum_{j=1}^m |\min\{p_j, U\} - \min\{p'_j, U\}|.$$

Observation 4.1. In the Dutch Walrasian mechanism, if there are zero copies of a good, letting its price be $+\infty$ will not affect the mechanism outcome.

Observation 4.2. Suppose bidders' demands satisfy the Gross Substitutes property. In both the English and Dutch Walrasian mechanisms, if $n_i \geq n'_i$, then $\mathbf{p}(n_i, n_{-i}) \preceq \mathbf{p}(n'_i, n_{-i})$, where $\mathbf{p} \preceq \mathbf{p}'$ means that, for all j , $p_j \leq p'_j$.

Definition 4.1. Given bidding profile (b_i, b_{-i}) , $\mathbf{n} = (n_j, n_{-j})$ is (ϵ, U) -bad for good j , if in the English Walrasian mechanism the distance between the prices is more than ϵ when an additional copy of good j is added to the market:

$$dist^U(\mathbf{p}^{Eng}((n_j, n_{-j}); (b_i, b_{-i})), \mathbf{p}^{Eng}((n_j + 1, n_{-j}); (b_i, b_{-i}))) > \epsilon.$$

Let $\mathbf{k} = (k, k, \dots, k)$ and $\mathbf{0} = (0, 0, \dots, 0)$ be l -vectors.

Definition 4.2. Given bidding profile \mathbf{b} , \mathbf{n} is (k, ϵ, U) -bad for good j if there is a vector \mathbf{n}' which is (ϵ, U) -bad for good j , such that $n'_h \leq n_h$ for all h , and $\sum_h n_h \leq k + \sum_h n'_h$. \vec{n} is (k, ϵ, U) -good if it is not (k, ϵ, U) -bad.

In Lemmas 4.1 and 4.2, we bound the number of (ϵ, U) -bad multiplicity vectors, and then in Lemma 4.3 we bound the probability of a (k, ϵ, U) -bad vector. Following this, in Lemma 4.4 and 4.5, assuming the multiplicity vector is $(k + 1, \epsilon, U)$ -good, we bound the difference between the English Walrasian mechanism prices and those of the Walrasian mechanism at hand. Next, in Lemma 4.6,

again for $(k+1, \epsilon, U)$ -good multiplicity vectors, we relate $u_i(\mathbf{x}_i(v_i, b_{-i}))$ to $v_i(\mathbf{x}_i(v_i, v_{-i}))$ and the prices paid; we then use this to carry out a PoA analysis. For brevity, we sometimes write $u_i(v_i, b_{-i})$ instead of $u_i(\mathbf{x}_i(v_i, b_{-i}))$.

Lemma 4.1. *In the English Walrasian mechanism, given n_{-j} and bidding profile \mathbf{b} , the number of values n_j for which (n_j, n_{-j}) is (ϵ, U) -bad for good j is at most $\frac{m}{\epsilon}U$.*

Proof. We prove the result by contradiction. Accordingly, let

$$S = \left\{ n_j \left| \text{dist}^U(\mathbf{p}^{Eng}((n_j, n_{-j}); \mathbf{b}), \mathbf{p}^{Eng}((n_j + 1, n_{-j}); \mathbf{b})) > \epsilon \right. \right\} \text{ and suppose that } |S| > \frac{m}{\epsilon}U.$$

$$\text{The proof uses a new function } pf(\cdot) : \quad pf(n_j) = \sum_{q=1}^m \min\{p_q^{Eng}((n_j, n_{-j}); \mathbf{b}), U\}.$$

$$\begin{aligned} \text{Then,} \quad \liminf_{n \rightarrow \infty} (pf(0) - pf(n)) &= \liminf_{n \rightarrow \infty} \sum_{h=0}^{n-1} (pf(h) - pf(h+1)) \\ &\geq \sum_{n_j \in S} (pf(n_j) - pf(n_j + 1)) > \frac{m}{\epsilon}U \cdot \epsilon = m \cdot U. \end{aligned} \quad (4.1)$$

The first inequality follows as by Observation 4.2, $pf(\cdot)$ is a non-increasing function. Further, by construction, $0 \leq pf(h) \leq l \cdot U$ for all h , thus $\liminf_{n \rightarrow \infty} (pf(0) - pf(n)) \leq l \cdot U$, contradicting (4.1). \square

Lemma 4.2. *In the English Walrasian mechanism with bidding profile \mathbf{b} , for a fixed n_{-j} , the number of values n_j for which (n_j, n_{-j}) is (k, ϵ, U) -bad for good j is at most $\frac{m}{\epsilon}U \cdot \binom{k+m}{m}$.*

Proof. Consider the case that $m \geq 2$. For (n_j, n_{-j}) to be (k, ϵ, U) -bad for good j we need an (ϵ, U) -bad vector $\mathbf{n}' \preceq \mathbf{n}$ for good j , with $\sum_{h \neq j} n_h - n'_h = c$ for some $0 \leq c \leq k$ and $n_j - n'_j \leq k - c$. There are $\binom{m-2+c}{c}$ ways of choosing the n'_{-j} . For each n'_{-j} , by Lemma 4.1, there are at most $\frac{m}{\epsilon}U$ points that are (ϵ, U) -bad for good j . For each (ϵ, U) -bad point, there are $k - c + 1$ choices for n_j . This gives a total of

$$\sum_{c=0}^k \frac{m}{\epsilon}U(k - c + 1) \binom{m-2+c}{c} = \frac{m}{\epsilon}U \sum_{c=0}^k \binom{m-1+c}{c} = \frac{m}{\epsilon}U \binom{m+k}{k}$$

(k, ϵ, U) -bad vectors. Note that the first equality follows by Lemma A.2 and the second equality follows by Lemma A.1.

For the case $m = 1$, for each (ϵ, U) -bad point for this good, it will cause at most $k + 1$ points to be (k, ϵ, U) -bad for this good. This gives a total of

$$\frac{m}{\epsilon}U(k + 1) = \frac{m}{\epsilon}U \binom{m+k}{k}.$$

(k, ϵ, U) -bad vectors. \square

For simplicity, let $\Lambda(m, k)$ denote $m \cdot \binom{k+m}{m}$.

Lemma 4.3. *In the English Walrasian mechanism with bidding profile \mathbf{b} , the probability that \mathbf{n} is (k, ϵ, U) -bad for some good or $\min_j n_j \leq k$ is at most*

$$m \cdot F(N) \left[\frac{U}{\epsilon} \Lambda(m, k) + k + 1 \right].$$

Proof. Conditioned on the bidding profile being \mathbf{b} ,

$$\begin{aligned} & \sum_{1 \leq j \leq m} \Pr[(\mathbf{n} \text{ is } (k, \epsilon, U)\text{-bad for good } j) \cup (n_j \leq k)] \\ & \leq \sum_{1 \leq j \leq m} \Pr[(\mathbf{n} \text{ is } (k, \epsilon, U)\text{-bad for good } j)] + \Pr[(n_j \leq k)] \\ & \leq \sum_{1 \leq j \leq m} \sum_{\bar{n}_{-j}} \left(\Pr[(\mathbf{n} \text{ is } (k, \epsilon, U)\text{-bad for good } j) | n_{-j} = \bar{n}_{-j}] \right. \\ & \quad \left. + \Pr[(n_j \leq k) | n_{-j} = \bar{n}_{-j}] \right) \cdot \Pr[n_{-j} = \bar{n}_{-j}] \\ & \leq mF(N) \left[\frac{m}{\epsilon} U \binom{k+m}{m} + k + 1 \right] \quad (\text{by Lemma 4.2}). \end{aligned}$$

□

Let $n_j^i(b_i, b_{-i})$ denote the number of copies of good j that bidder i receives with bidding profile (b_i, b_{-i}) and $\mathbf{n}^i(b_i, b_{-i})$ denote the corresponding vector. Also, let $p_j^{Eng}(\mathbf{n}; b_{-i})$ denote the market equilibrium prices when bidder i is not present.

Lemma 4.4. $p_j^{Eng}(\mathbf{n}; b_{-i}) \leq p_j(\mathbf{n}; (b_i, b_{-i})).$

Proof. Consider the situation with $\mathbf{n}' = \mathbf{n} - \mathbf{n}^i(b_i, b_{-i})$ and suppose that agent i is not present. Then $p_j(\mathbf{n}; (b_i, b_{-i}))$ is a market equilibrium.

$$\text{So } \forall j \quad p_j^{Eng}(\mathbf{n}'; b_{-i}) \leq p_j(\mathbf{n}; (b_i, b_{-i})).$$

$$\text{Since } \mathbf{n} \geq \mathbf{n}', \quad \forall j \quad p_j^{Eng}(\mathbf{n}; b_{-i}) \leq p_j^{Eng}(\mathbf{n}'; b_{-i}).$$

The lemma follows on combining these two inequalities. □

Lemma 4.5. *If \mathbf{n} is $(k+1, \epsilon, U)$ -good for all goods, and $n_j > k+1$ for all j , then*

$$\forall j \quad \min\{p_j(\mathbf{n}; (v_i, b_{-i})), U\} \leq \min\{p_j(\mathbf{n}; (b_i, b_{-i})), U\} + (k+1)\epsilon.$$

Proof. Let $\mathbf{d}^i \preceq \mathbf{n}^i(v_i, b_{-i})$ be a minimal set with $v_i(\mathbf{d}^i) = v_i(\mathbf{n}^i(v_i, b_{-i}))$. By Definition 2.3(i), $\sum_j d_j^i \leq k$. First, if $n_j^i(v_i, b_{-i}) > d_j^i$ then $p_j(\mathbf{n}; (v_i, b_{-i})) = 0$, as the pricing is given by a Walrasian Mechanism.

Consider the scenario with \mathbf{n}' copies of goods on offer, where for all j , $n'_j = n_j - d_j^i$ and suppose that bidder i is not present; then $\mathbf{p}(\mathbf{n}; (v_i, b_{-i}))$ is a market equilibrium.

So,

$$p_j(\mathbf{n}; (v_i, b_{-i})) \leq p_j^{Dut}(\mathbf{n}'; b_{-i}).$$

For all $j' \neq j$, let $n''_{j'} = n'_{j'}$ and let $n''_j = n'_j - 1$; then

$$p_j^{Dut}(\mathbf{n}'; b_{-i}) \leq p_j^{Eng}(\mathbf{n}''; b_{-i}),$$

and by Lemma 4.4,

$$p_j^{Eng}(\mathbf{n}''; b_{-i}) \leq p_j^{Eng}(\mathbf{n}''; b_i, b_{-i}).$$

As \mathbf{n} is $(k+1, \epsilon, U)$ -good for all goods, and as $\sum_h n_h - n''_h \leq k+1$, we conclude that

$$\begin{aligned} \min\{p_j(\mathbf{n}; (v_i, b_{-i})), U\} &\leq \min\{p_j^{Eng}(\mathbf{n}''; (b_i, b_{-i})), U\} \\ &\leq \min\{p_j^{Eng}(\mathbf{n}; (b_i, b_{-i})), U\} + (k+1)\epsilon \leq \min\{p_j(\mathbf{n}; (b_i, b_{-i})), U\} + (k+1)\epsilon. \end{aligned} \quad (4.2)$$

□

Let $|\mathbf{x}_i(\cdot)|$ denotes the total number of items in allocation \mathbf{x}_i . Let $\mathbf{d}_i \preceq \mathbf{x}_i(v_i, v_{-i})$ be a minimal set with $v_i(\mathbf{d}_i) = v_i(\mathbf{x}_i(v_i, v_{-i}))$. By Definition 2.3(i), $|\mathbf{d}_i| \leq k$.

Lemma 4.6. *If \mathbf{n} is $(k+1, \epsilon, U)$ -good with $U \geq v_i(s)$ for every single item s , $n_j > k+1$ for all j , v_i and b_i satisfy the gross substitutes property for all i , then*

$$u_i(v_i, b_{-i}) \geq v_i(\mathbf{x}_i(v_i, v_{-i})) - \sum_{s \in \mathbf{x}_i(v_i, v_{-i})} p_s(\mathbf{n}; (b_i, b_{-i})) - |\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i| \cdot (k+1)\epsilon,$$

where the sum is over all the items in allocation \mathbf{x}_i .

Proof. As we are using a Walrasian mechanism, for any allocation x'_i ,

$$v_i(x_i(v_i, b_{-i})) - \sum_{s \in \mathbf{x}_i(v_i, b_{-i})} p_s(\mathbf{n}; (v_i, b_{-i})) \geq v_i(\mathbf{x}'_i) - \sum_{s \in \mathbf{x}'_i} p_s(\mathbf{n}; (v_i, b_{-i})). \quad (4.3)$$

We let S denote the set of goods whose prices $p_s(\mathbf{n}; (v_i, b_{-i}))$ are larger than U . Then,

$$\begin{aligned} u_i(v_i, b_{-i}) &= v_i(\mathbf{x}_i(v_i, b_{-i})) - \sum_{s \in \mathbf{x}_i(v_i, b_{-i})} p_s(\mathbf{n}; (v_i, b_{-i})) \\ &\geq v_i((\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) \setminus S) - \sum_{s \in (\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) \setminus S} p_s(\mathbf{n}; (v_i, b_{-i})) \quad (\text{by (4.3)}) \end{aligned} \quad (4.4)$$

Since \mathbf{n} is $(k+1, \epsilon, U)$ -good, by Lemma 4.5,

$$\min\{p_s(\mathbf{n}; (v_i, b_{-i})), U\} \leq \min\{p_s(\mathbf{n}; (b_i, b_{-i})), U\} + (k+1)\epsilon.$$

$$\begin{aligned} \text{Therefore, for any } s \notin S, \quad p_s(\mathbf{n}; (v_i, b_{-i})) &\leq \min\{p_s(\mathbf{n}; (b_i, b_{-i})), U\} + (k+1)\epsilon \\ &\leq p_s(\mathbf{n}; (b_i, b_{-i})) + (k+1)\epsilon. \end{aligned}$$

$$\begin{aligned} \text{So,} \quad v_i((\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) \setminus S) - \sum_{s \in (\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) \setminus S} p_s(\mathbf{n}; (v_i, b_{-i})) \\ &\geq v_i((\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) \setminus S) - \sum_{s \in (\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) \setminus S} p_s(\mathbf{n}; (b_i, b_{-i})) \\ &\quad - |(\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) \setminus S| \cdot (k+1)\epsilon. \end{aligned} \quad (4.5)$$

For any $s \in S$, on applying Lemma 4.5, we obtain

$$U = \min\{p_s(\mathbf{n}; (v_i, b_{-i})), U\} \leq \min\{p_s(\mathbf{n}; (b_i, b_{-i})), U\} + (k+1)\epsilon,$$

which implies $p_s(\mathbf{n}; (b_i, b_{-i})) + (k+1)\epsilon \geq U$. Also,

$$v_i(\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) - v_i((\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) \setminus S) \leq v_i((\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) \cap S) \leq |(\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) \cap S| \cdot U,$$

where the first inequality follows by the Gross Substitutes assumption, and the second by Gross Substitutes and because by assumption $v_i(s) \leq U$ for all single items s . Thus,

$$\begin{aligned}
& v_i((\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) \setminus S) - \sum_{s \in (\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) \setminus S} p_s(\mathbf{n}; (b_i, b_{-i})) - |(\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) \setminus S| \cdot (k+1)\epsilon \\
& \geq v_i(\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) - |(\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) \cap S| \cdot U \\
& \quad - \sum_{s \in (\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) \setminus S} p_s(\mathbf{n}; (b_i, b_{-i})) - |(\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) \setminus S| \cdot (k+1)\epsilon \\
& \geq v_i(\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i) - \sum_{s \in \mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i} p_s(\mathbf{n}; (b_i, b_{-i})) - |\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i| \cdot (k+1)\epsilon \\
& \geq v_i(\mathbf{x}_i(v_i, v_{-i})) - \sum_{s \in \mathbf{x}_i(v_i, v_{-i})} p_s(\mathbf{n}; (b_i, b_{-i})) - |\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i| \cdot (k+1)\epsilon.
\end{aligned} \tag{4.6}$$

By (4.4), (4.5) and (4.6),

$$u_i(v_i, b_{-i}) \geq v_i(\mathbf{x}_i(v_i, v_{-i})) - \sum_{s \in \mathbf{x}_i(v_i, v_{-i})} p_s(\mathbf{n}; (b_i, b_{-i})) - |\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i| \cdot (k+1)\epsilon.$$

□

Lemma 4.7. *If Assumption 3.1 holds, then $\mathbb{E}_{v_i}[\max_s\{v_i(s)\}] < m \cdot \zeta$.*

Proof. $\mathbb{E}_{v_i}[\max_s\{v_i(s)\}] \leq \mathbb{E}_{v_i}[\sum_s v_i(s)] \leq \sum_s \mathbb{E}_{v_i}[v_i(s)] \leq m \cdot \zeta$. □

Here we prove a slightly weaker version of Theorem 3.1 which demonstrates the main ideas. The proof of Theorem 3.1 can be found in appendix.

Theorem 4.1. *In a large Walrasian auction which satisfies Assumptions 3.1 and 3.2 and with buyers whose valuation and bid functions are monotone and satisfy the gross substitutes property,*

$$\text{SW(NE)} \geq \left(1 - \frac{3k \cdot \zeta \cdot m}{\rho} \sqrt{(k+2)mF(N)\Lambda(m, k+1)}\right) \text{SW(OPT)}.$$

Proof. By Lemma 4.6, if \mathbf{n} is $(k+1, \epsilon \cdot \max_s\{v_i(s)\}, \max_s\{v_i(s)\})$ -good and $n_j > k+1$ for all j , then

$$\begin{aligned}
u_i(v_i, b_{-i}) & \geq v_i(\mathbf{x}_i(v_i, v_{-i})) - \sum_{s \in \mathbf{x}_i(v_i, v_{-i})} p_s(\mathbf{n}; (b_i, b_{-i})) \\
& \quad - |\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i| \cdot (k+1)\epsilon \cdot \max_s\{v_i(s)\}.
\end{aligned}$$

By Lemma 4.3, the probability that \mathbf{n} is $(k+1, \epsilon \cdot \max_s\{v_i(s)\}, \max_s\{v_i(s)\})$ -bad or $n_j \leq k+1$ for some j is less than

$$mF(N) \left[\frac{1}{\epsilon} \Lambda(m, k+1) + k+2 \right],$$

$$\text{and } \mathbb{E}_{\mathbf{n}}[u_i(v_i, b_{-i})] \geq \mathbb{E}_{\mathbf{n}} \left[v_i(\mathbf{x}_i(v_i, v_{-i})) - \sum_{s \in \mathbf{x}_i(v_i, v_{-i})} p_s(\mathbf{n}; (b_i, b_{-i})) \right. \\ \left. - |\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i| \cdot (k+1)\epsilon \cdot \max_s \{v_i(s)\} \right] \\ - mF(N) \left[\frac{1}{\epsilon} \Lambda(m, k+1) + k+2 \right] \cdot k \cdot \max_s \{v_i(s)\}.$$

Here, the expectation is taken over the randomness on the multiplicities of the goods; the inequality holds since $u_i(v_i, b_{-i}) \geq 0$ and $v_i(\mathbf{x}_i(v_i, v_{-i})) \leq k \cdot \max_s \{v_i(s)\}$.

Taking the expectation over the valuation of agent i yields

$$\mathbb{E}_{v_i}[\mathbb{E}_{\mathbf{n}}[u_i(v_i, b_{-i})]] \geq \mathbb{E}_{v_i} \left[\mathbb{E}_{\mathbf{n}} \left[v_i(\mathbf{x}_i(v_i, v_{-i})) - \sum_{s \in \mathbf{x}_i(v_i, v_{-i})} p_s(\mathbf{n}; (b_i, b_{-i})) \right. \right. \\ \left. \left. - |\mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i| \cdot (k+1)\epsilon \cdot \max_s \{v_i(s)\} \right] \right. \\ \left. - mF(N) \left[\frac{1}{\epsilon} \Lambda(m, k+1) + k+2 \right] \cdot k \cdot \max_s \{v_i(s)\} \right] \\ \geq \mathbb{E}_{v_i} \left[\mathbb{E}_{\mathbf{n}} \left[v_i(\mathbf{x}_i(v_i, v_{-i})) - \sum_{s \in \mathbf{x}_i(v_i, v_{-i})} p_s(\mathbf{n}; (b_i, b_{-i})) \right] \right] \\ - \mathbb{E}_{v_i}[\max_s \{v_i(s)\}] \cdot k \cdot (k+1)\epsilon \\ - \mathbb{E}_{v_i}[\max_s \{v_i(s)\}] mF(N) \left[\Lambda(m, k+1) \frac{k}{\epsilon} + (k+2)k \right].$$

By Lemma 4.7, $\mathbb{E}_{v_i}[\max_s \{v_i(s)\}] \leq m \cdot \zeta$. Thus

$$\mathbb{E}_{v_i}[\mathbb{E}_{\mathbf{n}}[u_i(v_i, b_{-i})]] \geq \mathbb{E}_{v_i} \left[\mathbb{E}_{\mathbf{n}} \left[v_i(\mathbf{x}_i(v_i, v_{-i})) - \sum_{s \in \mathbf{x}_i(v_i, v_{-i})} p_s(\mathbf{n}; (b_i, b_{-i})) \right] \right] \\ - \zeta \cdot m \cdot k(k+1)\epsilon \\ - \zeta \cdot m \cdot m \cdot F(N) \left[\Lambda(m, k+1) \frac{k}{\epsilon} + (k+2)k \right].$$

Let $R(\mathbf{b})$ denote the revenue when the bidding profile is \mathbf{b} . By Assumption 3.2, the optimal welfare $SW(OPT) > \rho N$. Now, summing over all the bidders yields

$$\sum_i \mathbb{E}_{\mathbf{v}, \mathbf{b}, \mathbf{n}}[u_i(v_i, b_{-i})] \geq \sum_i \mathbb{E}_{\mathbf{v}, \mathbf{b}} \left[\mathbb{E}_{\mathbf{n}} \left[v_i(\mathbf{x}_i(v_i, v_{-i})) - \sum_{s \in \mathbf{x}_i(v_i, v_{-i})} p_s(\mathbf{n}; (b_i, b_{-i})) \right] \right] \\ - \zeta \cdot m \cdot m \cdot F(N) \left[\Lambda(m, k+1) \frac{k}{\epsilon} + (k+2)k \right] \cdot N \\ - \zeta \cdot m \cdot k(k+1)\epsilon \cdot N \\ \geq \left(1 - \frac{\zeta \cdot m \cdot m \cdot F(N) \left[\Lambda(m, k+1) \frac{k}{\epsilon} + (k+2)k \right]}{\rho} \right. \\ \left. - \frac{\zeta \cdot m \cdot k(k+1)\epsilon}{\rho} \right) SW(OPT) - \mathbb{E}_{\mathbf{b}}[R(b_i, b_{-i})].$$

Using the smooth technique for Bayesian settings [3],

$$\begin{aligned} \text{SW(NE)} &\geq \left(1 - \frac{\zeta \cdot m \cdot m \cdot F(N) [\Lambda(m, k+1) \frac{k}{\epsilon} + (k+2)k]}{\rho} \right. \\ &\quad \left. - \frac{\zeta \cdot m \cdot k(k+1)\epsilon}{\rho} \right) \text{SW(OPT)}. \end{aligned}$$

Now set $\epsilon = \sqrt{\frac{m \cdot F(N) \Lambda(m, k+1)}{k+1}}$. The claimed bound follows. \square

5 Large Fisher Markets

Theorem 3.3 will follow from the following lemma.

Lemma 5.1. *For any bidding profile \mathbf{b} and any value profile \mathbf{v} which are homogeneous of degree 1, concave, continuous, monotone and which satisfy the gross substitutes property,*

$$\sum_{i=1}^n e_i \cdot \log(u_i(v_i, b_{-i})) \geq \sum_{i=1}^n e_i \cdot \log(u_i(v_i, v_{-i})) - m \cdot \max_i e_i.$$

Proof of Theorem 3.3: On exponentiating the expressions on both sides in the statement of Lemma 5.1 we obtain

$$\prod_i u_i(v_i, b_{-i})^{e_i} \geq \frac{1}{e^{m \cdot \max_i e_i}} \prod_i u_i(v_i, v_{-i})^{e_i}.$$

Therefore,

$$\prod_i \left(\frac{u_i(v_i, b_{-i})}{u_i(v_i, v_{-i})} \right)^{e_i} \geq \frac{1}{e^{m \cdot \max_i e_i}}.$$

Using the weighted GM-AM inequality, we obtain

$$\frac{\sum_i e_i \frac{u_i(v_i, b_{-i})}{u_i(v_i, v_{-i})}}{\sum_i e_i} \geq \left(\prod_i \left(\frac{u_i(v_i, b_{-i})}{u_i(v_i, v_{-i})} \right)^{e_i} \right)^{\frac{1}{\sum_i e_i}} \geq \left(\frac{1}{e^{m \cdot \max_i e_i}} \right)^{\frac{1}{\sum_i e_i}} = e^{-\frac{m \cdot \max_i e_i}{\sum_i e_i}}.$$

Since for all i , $u_i(v_i, v_{-i}) = t e_i$,

$$\sum_i u_i(v_i, b_{-i}) \geq e^{-\frac{m \cdot \max_i e_i}{\sum_i e_i}} \sum_i u_i(v_i, v_{-i}).$$

The theorem follows on applying the smooth technique. \square

To prove Lemma 5.1, we need the following claim; intuitively, it states that a single bidder can cause the prices to change by only a small amount.

Lemma 5.2.

$$\mathbf{p}(v_i, b_{-i}) \leq \mathbf{p}(b_i, b_{-i}) + \max_i e_i \cdot 1$$

Proof of Lemma 5.1: Consider the dual of the Eishenberg-Gale convex program:

$$\begin{aligned} \min_p \max_x \sum_{i=1}^n e_i \cdot \log(u_i(x_{i1}, x_{i2}, \dots, x_{im})) - \sum_{i,j} p_j x_{ij} + \sum_j p_j \\ \text{s.t.} \quad \forall j : \quad p_j \geq 0 \\ \forall i, j : \quad x_{ij} \geq 0. \end{aligned}$$

Let \mathbf{p} denote an arbitrary collection of prices, and \mathbf{p}^* denote the prices with truthful bids. Since \mathbf{p}^* minimizes the dual program,

$$\begin{aligned} \max_{\mathbf{x} \geq 0} \quad & \sum_{i=1}^n e_i \cdot \log(u_i(x_{i1}, x_{i2}, \dots, x_{im})) - \sum_{i,j} p_j x_{ij} + \sum_j p_j \\ & \geq \max_{\mathbf{x} \geq 0} \quad \sum_{i=1}^n e_i \cdot \log(u_i(x_{i1}, x_{i2}, \dots, x_{im})) - \sum_{i,j} p_j^* x_{ij} + \sum_j p_j^*. \end{aligned} \quad (5.1)$$

Let x_{ij}^* be an allocation over all goods j and bidders i at prices p that maximize (5.1). As u_i is homogeneous of degree 1, u_i is differentiable in the direction \mathbf{x}_i . It follows that

$$\lim_{\epsilon \rightarrow 0} \frac{[e_i \cdot \log u_i((1 + \epsilon)\mathbf{x}_i^*) - \sum_j p_j(1 + \epsilon)x_{ij}^*] - [e_i \cdot \log u_i(\mathbf{x}_i^*) - \sum_j p_j x_{ij}^*]}{\epsilon} = 0. \quad (5.2)$$

The LHS of (5.2) equals $e_i - \sum_j p_j x_{ij}^*$, implying that

$$e_i = \sum_j p_j x_{ij}^*.$$

Therefore,

$$\begin{aligned} \max_{\mathbf{x}: \forall i \sum x_{ij} p_j = e_i} \quad & \sum_{i=1}^n e_i \cdot \log(u_i(x_{i1}, x_{i2}, \dots, x_{im})) + \sum_j p_j \\ & \geq \max_{\mathbf{x}: \forall i \sum x_{ij} p_j^* = e_i} \quad \sum_{i=1}^n e_i \cdot \log(u_i(x_{i1}, x_{i2}, \dots, x_{im})) + \sum_j p_j^*. \end{aligned} \quad (5.3)$$

If all the prices stay the same or increase, a buyer's optimal utility stays the same or reduces.

Using the price upper bound from Lemma 5.2, it follows that

$$\begin{aligned}
\sum_{i=1}^n e_i \cdot \log(u_i(v_i, b_{-i})) &\geq \sum_{i=1}^n \max_{\mathbf{x}: \forall i \sum x_{ij} (p_j(b_i, b_{-i}) + \max_{i' \neq i} e_{i'}) = e_i} e_i \cdot \log(u_i(x_{i1}, x_{i2}, \dots, x_{im})) \\
&= \sum_{i=1}^n \max_{\mathbf{x}: \forall i \sum x_{ij} (p_j(b_i, b_{-i}) + \max_{i' \neq i} e_{i'}) = e_i} e_i \cdot \log(u_i(x_{i1}, x_{i2}, \dots, x_{im})) \\
&\quad + \sum_j (p_j(b_i, b_{-i}) + \max_{i' \neq i} e_{i'}) - \sum_j (p_j(b_i, b_{-i}) + \max_{i' \neq i} e_{i'}) \\
&\geq \sum_{i=1}^n \max_{\mathbf{x}: \forall i \sum x_{ij} p_j^* = e_i} e_i \cdot \log(u_i(x_{i1}, x_{i2}, \dots, x_{im})) \\
&\quad + \sum_j p_j^* - \sum_j (p_j(b_i, b_{-i}) + \max_{i' \neq i} e_{i'}) \quad \text{by (5.3)} \\
&\geq \sum_{i=1}^n \max_{\mathbf{x}: \forall i \sum_j x_{ij} p_j^* = e_i} e_i \cdot \log(u_i(x_{i1}, x_{i2}, \dots, x_{im})) - m \max_i e_i \\
&\quad \text{as } \sum_j p_j^* = \sum_i e_i = \sum_j p_j(b_i, b_{-i}) \\
&= \sum_{i=1}^n e_i \log(u_i(v_i, b_{-i})) - m \max_i e_i.
\end{aligned}$$

□

The proof of Lemma 5.2 uses the following notation and follows from Lemmas 5.3 and 5.4 below. \mathbf{p} denotes the prices when the i th bidder is not participating and the bidding profile is b_{-i} ; \mathbf{x} denotes the resulting allocation. Similarly, $\hat{\mathbf{p}}$ denotes the prices when the bidding profile is (b_i, b_{-i}) ; $\hat{\mathbf{x}}$ denotes the resulting allocation.

Lemma 5.3. $\mathbf{p} \preceq \hat{\mathbf{p}}$.

Lemma 5.4. $\hat{\mathbf{p}} \preceq \mathbf{p} + e_i \cdot \mathbf{1}$.

Proof of Lemma 5.2: By Lemmas 5.3 and 5.4, the prices are lower bounded by prices \mathbf{p} and upper bounded by prices $\mathbf{p} + e_i \cdot \mathbf{1}$. So $\mathbf{p}(v_i, b_{-i}) \leq \mathbf{p} + e_i \cdot \mathbf{1} \leq \mathbf{p}(b_i, b_{-i}) + e_i \cdot \mathbf{1} \leq \mathbf{p}(b_i, b_{-i}) + \max_i e_i \cdot \mathbf{1}$. □

Lemma 5.4 follows readily from Lemma 5.3.

Proof of Lemma 5.4: Since $\mathbf{1} \cdot \mathbf{p} + e_i = \mathbf{1} \cdot \hat{\mathbf{p}}$ and $\mathbf{p} \preceq \hat{\mathbf{p}}$, the lemma follows. □

We finish by proving that Lemma 5.3 holds in two scenarios: single-demand WGS utility functions and linear utility functions.

5.1 Single-Demand WGS Utility Functions

Proof of Lemma 5.3: For a contradiction, we suppose there is an item j such that $p_j > \hat{p}_j$.

Let ϵ be a very small constant such that $\epsilon < p_k$ for all $p_k \neq 0$ and $\epsilon < \hat{p}_k$ for all $\hat{p}_k \neq 0$.

Let \mathbf{p}' denote the following collection of prices: $p'_k = p_k$ if $p_k \neq 0$, and $p'_k = \epsilon$ otherwise. We consider the resulting demands for a bidder $h \neq i$. Recall that \mathbf{x}_h denotes bidder h 's demand at

prices \mathbf{p} . \mathbf{x}'_h will denote her demand at prices \mathbf{p}' . By the WGS property, $x'_{hk} = x_{hk}$ if $p_k \neq 0$, and $x'_{hk} = 0$ if $p_k = 0$, i.e. if $p'_k = \epsilon$.⁵

Analogously, let $\hat{p}'_k = \hat{p}_k$ if $\hat{p}_k \neq 0$, and $\hat{p}'_k = \epsilon$ otherwise. Let $\hat{\mathbf{x}}_h$ denote bidder h 's demand at prices $\hat{\mathbf{p}}$, and $\hat{\mathbf{x}}'_h$ her demand at prices $\hat{\mathbf{p}}'$. Again, $\hat{x}'_{hk} = \hat{x}_{hk}$ if $\hat{p}_k \neq 0$, and $\hat{x}'_{hk} = 0$ if $\hat{p}'_k = \epsilon$.

Now, we look at the items l which have the smallest ratio between p'_l and p'_l .

$$S = \left\{ l \mid \frac{\hat{p}'_l}{p'_l} = \min_k \frac{\hat{p}'_k}{p'_k} \right\}.$$

By assumption, $p_j > \hat{p}_j$; therefore $p'_j > \hat{p}'_j$. Thus, for $l \in S$, $\frac{\hat{p}'_l}{p'_l} < 1$. For simplicity, let η denote this ratio. Note that this inequality implies $p'_l > \epsilon$, and thus $p_l = p'_l > 0$. Also,

$$p_l = p'_l > \hat{p}'_l > 0. \quad (5.4)$$

We now consider the following procedure:

First multiply \mathbf{p}' by η . By the homogeneity of the utility function, bidder h 's demand at prices $\eta \cdot \mathbf{p}'$ will be $\frac{1}{\eta} \mathbf{x}'_h$. Note that $\eta \cdot p'_l = \hat{p}'_l$ for any $l \in S$ and $\eta \cdot p'_k < \hat{p}'_k$ for any $k \notin S$.

Second, increase the prices of $\eta \cdot \mathbf{p}'$ to $\hat{\mathbf{p}}'$. Since for $l \in S$ the two prices are the same, by the Gross Substitutes property, $\hat{x}'_{hl} \geq \frac{1}{\eta} x'_{hl}$ for any $l \in S$.

Summing over all the bidders except i ,

$$\sum_{h \neq i} \hat{x}'_{hl} \geq \frac{1}{\eta} \sum_{h \neq i} x'_{hl} \quad \text{for } l \in S.$$

By (5.4), $p_l > 0$ for any $l \in S$; hence $\sum_{h \neq i} x'_{hl} = \sum_{h \neq i} x_{hl} = 1$. So, since $\eta < 1$,

$$\sum_{h \neq i} \hat{x}'_{hl} > \sum_{h \neq i} x'_{hl} = \sum_{h \neq i} x_{hl} = 1 \quad \text{for } l \in S. \quad (5.5)$$

For all h and l , $\hat{x}_{hl} \geq \hat{x}'_{hl}$. Therefore,

$$\sum_h \hat{x}_{hl} \geq \sum_{h \neq i} \hat{x}'_{hl} > \sum_{h \neq i} x_{hl} = 1 \quad \text{for } l \in S.$$

As $\sum_h \hat{x}_{hl} \leq 1$, this is impossible and yields a contradiction. \square

5.2 Linear Utility function

Proof of Lemma 5.3: For a contradiction, we suppose there is an item j such that $p_j > \hat{p}_j$. Now, we look at the items j which have the smallest ratio between p_l and \hat{p}_l .

$$S = \left\{ l \mid \frac{\hat{p}_l}{p_l} = \min_k \frac{\hat{p}_k}{p_k} \right\}.$$

For simplicity, we set $\frac{0}{x} = 0$ for $x > 0$, $\frac{0}{0} = 1$ and $\frac{x}{0} = +\infty$ for $x > 0$.

For linear utility functions, we use the following observation: if at prices \mathbf{p} a bidder's favorite items include some items in S , then at prices $\hat{\mathbf{p}}$ her favorite items will all be in S .

⁵Changing the prices from \mathbf{p} to \mathbf{p}' , one by one, by setting p'_k to ϵ , which happens when $p_k = 0$, only increases the demand for other goods, but as no spending is released by this price increase, these demands are in fact unchanged.

Therefore, as the price of each good equals the total spending on that good,

$$\sum_{l \in S} p_l \leq \sum_{l \in S} \hat{p}'_l.$$

This implies that $\min_k \frac{\hat{p}_k}{p_k} = 1$, and the lemma follows. \square

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A Omitted Proofs

A.1 Proofs from Section 3

Proof of Lemma 3.1: Let $\#\text{items}_j$ denote the number of copies of good j that are present, and let N_j denote the number of buyers for which good j has the largest expected value (breaking ties arbitrarily). By Chebyshev's Theorem, $\Pr[\#\text{items}_j > \mathbb{E}[\#\text{items}_j] - t \cdot \Gamma(\#\text{items}_j)] \geq 1 - \frac{1}{t^2}$. We set t equal to $1 + \lambda$, where λ is the parameter in Assumption 3.3. Then by Assumption 3.3, $\Pr[\#\text{items}_j > \lambda^2 \cdot \mathbb{E}[\#\text{items}_j]] \geq \frac{2\lambda + \lambda^2}{(1+\lambda)^2}$, which implies $\Pr[\#\text{items}_j > \lambda^2 \alpha N] \geq \frac{2\lambda + \lambda^2}{(1+\lambda)^2}$. If at least $\lambda^2 \alpha N$ copies of good j are available, then by Assumption 3.4, there is an assignment with valuation at least $\rho' \cdot \min\{N_j, \lambda^2 \alpha N\}$. Therefore, the social welfare is at least $\sum_j \min\{N_j, \lambda^2 \alpha N\} \frac{2\lambda + \lambda^2}{(1+\lambda)^2} \cdot \rho' \geq \lambda^2 \alpha \frac{2\lambda + \lambda^2}{(1+\lambda)^2} N \cdot \rho'$. \square

A.2 Proofs from Section 4

Lemma A.1.

$$\binom{m+n-1}{n} = \sum_{i=0}^n \binom{m+i-2}{i}$$

Lemma A.2.

$$\sum_{n=0}^k \binom{m+n-1}{n} = \sum_{n=0}^k (k-n+1) \binom{m+n-2}{n}.$$

Proof.

$$\begin{aligned} \sum_{n=0}^k \binom{m+n-1}{n} &= \sum_{n=0}^k \sum_{i=0}^n \binom{m+i-2}{i} = \sum_{i=0}^k \sum_{n=i}^k \binom{m+i-2}{i} \\ &= \sum_{i=0}^k (k-i+1) \binom{m+i-2}{i}. \end{aligned}$$

\square

Proof of Theorem 3.1: By Lemma 4.3, the probability that \mathbf{n} is $(k+1, \frac{\max_s\{v_i(s)\}}{2^c}, \max_s\{v_i(s)\})$ -bad or $n_j \leq k+1$ for some j is less than

$$\begin{aligned} mF(N) &\left[\frac{1}{\frac{\max_s\{v_i(s)\}}{2^c}} \max_s\{v_i(s)\} \Lambda(m, k+1) + k+2 \right] \\ &= mF(N) [2^c \Lambda(m, k+1) + k+2]. \end{aligned}$$

So, for any integer c' ,

$$\begin{aligned}
\mathbb{E}_{\mathbf{n}}[u_i(v_i, b_{-i})] &\geq \mathbb{E}_{\mathbf{n}} \left[v_i(\mathbf{x}_i(v_i, v_{-i})) - \sum_{s \in \mathbf{x}_i(v_i, v_{-i})} p_s(\mathbf{n}; (b_i, b_{-i})) \right. \\
&\quad - \sum_{c=1}^{c'} \mathbb{1} \left[\mathbf{n} \text{ is } (k+1, \frac{\max_s \{v_i(s)\}}{2^c}, \max_s \{v_i(s)\})\text{-bad and} \right. \\
&\quad \quad \left. \left. (k+1, \frac{\max_s \{v_i(s)\}}{2^{c-1}}, \max_s \{v_i(s)\})\text{-good} \right] \right. \\
&\quad \cdot \left| \mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i \right| \cdot (k+1) \frac{\max_s \{v_i(s)\}}{2^{c-1}} \\
&\quad - \mathbb{1} \left[\mathbf{n} \text{ is } (k+1, \frac{\max_s \{v_i(s)\}}{2^{c'-1}}, \max_s \{v_i(s)\})\text{-good} \right] \\
&\quad \cdot \left| \mathbf{x}_i(v_i, v_{-i}) \cap \mathbf{d}_i \right| \cdot (k+1) \frac{\max_s \{v_i(s)\}}{2^{c'}} \left. \right] \\
&\geq \mathbb{E}_{\mathbf{n}} \left[v_i(\mathbf{x}_i(v_i, v_{-i})) - \sum_{s \in \mathbf{x}_i(v_i, v_{-i})} p_s(\mathbf{n}; (b_i, b_{-i})) \right] \\
&\quad - \sum_{c=1}^{c'} mF(N) [2^c \Lambda(m, k+1) + k+2] \cdot k \cdot (k+1) \frac{\max_s \{v_i(s)\}}{2^{c-1}} \\
&\quad - k \cdot (k+1) \frac{\max_s \{v_i(s)\}}{2^{c'}} \\
&\geq \mathbb{E}_{\mathbf{n}} \left[v_i(\mathbf{x}_i(v_i, v_{-i})) - \sum_{s \in \mathbf{x}_i(v_i, v_{-i})} p_s(\mathbf{n}; (b_i, b_{-i})) \right. \\
&\quad \left. - c' \cdot mF(N) [2\Lambda(m, k+1) + k+2] \cdot k \cdot (k+1) \cdot \max_s \{v_i(s)\} \right. \\
&\quad \left. - k \cdot (k+1) \frac{\max_s \{v_i(s)\}}{2^{c'}} \right].
\end{aligned}$$

Summing over all the bidders and integrating w.r.t. \mathbf{v} and \mathbf{b} gives

$$\begin{aligned}
\sum_i \mathbb{E}_{\mathbf{v}, \mathbf{b}, \mathbf{n}}[u_i(v_i, b_{-i})] &\geq \text{SW(OPT)} - \mathbb{E}_{\mathbf{b}}[\text{R}(b_i, b_{-i})] \\
&\quad - N \cdot c' \cdot mF(N) [2\Lambda(l, k+1) + k+2] \cdot k \cdot (k+1) \cdot \zeta \cdot m \\
&\quad - N \cdot k \cdot (k+1) \frac{1}{2^{c'}} \cdot \zeta \cdot m.
\end{aligned}$$

Using the smooth technique for Bayesian settings [3] yields

$$\begin{aligned}
\text{SW(NE)} &\geq \left(1 - \frac{\zeta \cdot m \cdot k \cdot (k+1) \frac{1}{2^{c'}}}{\rho} \right. \\
&\quad \left. - \frac{\zeta \cdot m \cdot c' \cdot mF(N) [2\Lambda(m, k+1) + k+2] \cdot k \cdot (k+1)}{\rho} \right) \text{SW(OPT)}.
\end{aligned}$$

Let $Y = mF(N)[2\Lambda(m, k+1)]$. Set $c' = \lceil \log_2 \frac{1}{Y} - \log_2 \log_2 \frac{1}{Y} \rceil$; then $\frac{1}{2^{c'}} \leq Y \log_2 \frac{1}{Y}$. So,

$$\text{SW(NE)} \geq \left(1 - \frac{3 \cdot k \cdot (k+1) \cdot \zeta \cdot m}{\rho} \cdot Y \cdot \lceil \log_2 \frac{1}{Y} \rceil\right) \text{SW(OPT)}.$$

□

B Regret Minimization

B.1 Walrasian Market

We note the following corollary to Theorem 3.1.

Corollary B.1. *In a large Walrasian auction which satisfies Assumptions 3.1 and 3.2, if v_i and b_i are monotone and satisfy the gross substitutes property for all i , then*

$$\sum_i \mathbb{E}_{\mathbf{n}, \mathbf{v}, \mathbf{b}}[u_i(v_i, b_{-i})] \geq \left(1 - \frac{3 \cdot k \cdot (k+1) \cdot \zeta \cdot m}{\rho} \cdot Y \cdot \lceil \log_2 \frac{1}{Y} \rceil\right) \text{SW(OPT)} - \mathbb{E}_{\mathbf{b}}[R(b_i, b_{-i})]$$

where $Y = m \cdot F(N) \left[2m \binom{k+1+m}{m}\right]$.

Proof of Theorem 3.2: Since player i uses a regret minimizing algorithm and she is a (γ, δ) -player,

$$\mathbb{E}_{\mathbf{n}} \left[\sum_{t=1}^T v_i(b_i^t, b_{-i}^t) \right] \geq \mathbb{E}_{\mathbf{n}} \left[\sum_{t=1}^T u_i(v_i, b_{-i}^t) - \Phi(|K_i|, T) \cdot \left(\max_{\mathbf{x}_i} v_i(\mathbf{x}_i) \cdot \gamma + \delta \right) \right].$$

Summing over all bidders and integrating w.r.t. \mathbf{v} and \mathbf{b} gives

$$\begin{aligned} \mathbb{E}_{\mathbf{n}, \mathbf{v}, \mathbf{b}} \left[\sum_i \sum_{t=1}^T u_i(b_i^t, b_{-i}^t) \right] &\geq \mathbb{E}_{\mathbf{n}, \mathbf{v}, \mathbf{b}} \left[\sum_i \sum_{t=1}^T u_i(v_i, b_{-i}^t) - \Phi(|K_i|, T) \cdot \left(\max_{\mathbf{x}_i} v_i(\mathbf{x}_i) \cdot \gamma + \delta \right) \right] \\ &\geq \mathbb{E}_{\mathbf{n}, \mathbf{v}, \mathbf{b}} \left[\sum_i \sum_{t=1}^T u_i(v_i, b_{-i}^t) - \Phi(|K_i|, T) \cdot (km\zeta\gamma + \delta) \right]. \end{aligned}$$

By Corollary B.1,

$$\begin{aligned} \sum_i \mathbb{E}_{\mathbf{n}, \mathbf{v}, \mathbf{b}}[u_i(v_i, b_{-i})] &\geq \left(1 - \frac{3 \cdot k \cdot (k+1) \cdot \zeta \cdot m}{\rho} \cdot Y \cdot \lceil \log_2 \frac{1}{Y} \rceil\right) \text{SW(OPT)} \\ &\quad - \mathbb{E}_{\mathbf{b}}[R(b_i, b_{-i})]. \end{aligned} \tag{B.1}$$

Therefore, since valuation equals utility plus payment,

$$\begin{aligned}
& \mathbb{E}_{\mathbf{n}, \mathbf{v}, \mathbf{b}} \left[\frac{1}{T} \sum_i \sum_{t=1}^T v_i(\mathbf{x}_i(b_i^t, b_{-i}^t)) \right] \\
&= \mathbb{E}_{\mathbf{n}, \mathbf{v}, \mathbf{b}} \left[\frac{1}{T} \sum_i \sum_{t=1}^T (u_i(b_i^t, b_{-i}^t) + R(b_i^t, b_{-i}^t)) \right] \\
&\geq \frac{1}{T} \mathbb{E}_{\mathbf{n}, \mathbf{v}, \mathbf{b}} \left[\sum_i \left(\sum_{t=1}^T (u_i(v_i, b_{-i}^t) + R(b_i^t, b_{-i}^t)) - \Phi(|K_i|, T) \cdot (km\zeta\gamma + \delta) \right) \right] \\
&\geq \left(1 - \frac{3 \cdot k \cdot (k+1) \cdot \zeta \cdot m}{\rho} \cdot Y \cdot \lceil \log_2 \frac{1}{Y} \rceil \right) \text{SW(OPT)} \\
&\quad - \frac{1}{T} \sum_i \Phi(|K_i|, T) \cdot (km\zeta\gamma + \delta) \quad \text{by (B.1)} \\
&\geq \left(1 - \frac{3 \cdot k \cdot (k+1) \cdot \zeta \cdot m}{\rho} \cdot Y \cdot \lceil \log_2 \frac{1}{Y} \rceil \right) \text{SW(OPT)} \\
&\quad - \max_i \Phi(|K_i|, T) \cdot (km\zeta\gamma + \delta) \frac{1}{\rho \cdot T} \text{SW(OPT)} \\
&= \left(1 - \frac{3 \cdot k \cdot (k+1) \cdot \zeta \cdot m}{\rho} \cdot Y \cdot \lceil \log_2 \frac{1}{Y} \rceil \right. \\
&\quad \left. - \frac{\max_i \Phi(|K_i|, T) \cdot (km\zeta\gamma + \delta)}{\rho \cdot T} \right) \text{SW(OPT)}.
\end{aligned}$$

□

B.2 Fisher Market with Reserve Prices

Theorem 3.4 will follow from the following lemma; its proof is given in Appendix C.

Theorem B.1. *For any bidding profile \mathbf{b} and any value profile \mathbf{v} which are homogeneous of degree 1, concave, continuous, monotone and gross substitutes, if the reserve prices $r_j \leq \frac{1}{4}p_j^*$ for any j , then*

$$\sum_i u_i(v_i, b_{-i}) \geq e^{-\frac{2m}{5L}} \sum_i u_i(\mathbf{x}_i(\mathbf{p}^*)).$$

Proof of Theorem 3.4: Since player i uses a regret minimizing algorithm and the maximal payoff is $\lambda u_i(v_i, v_{-i})$,

$$\sum_{t=1}^T u_i(b_i^t, b_{-i}^t) \geq \sum_{t=1}^T u_i(v_i, b_{-i}^t) - \Phi(|K_i|, T) \cdot \lambda u_i(v_i, v_{-i}).$$

Summing over all the bidders gives

$$\begin{aligned}
\sum_i \sum_{t=1}^T u_i(b_i^t, b_{-i}^t) &\geq \sum_i \sum_{t=1}^T u_i(v_i, b_{-i}^t) - \sum_i \Phi(|K_i|, T) \cdot \lambda u_i(v_i, v_{-i}) \\
&\geq \sum_i \sum_{t=1}^T u_i(v_i, b_{-i}^t) - \sum_i \max_{i'} \Phi(|K_{i'}|, T) \cdot \lambda u_i(v_i, v_{-i}).
\end{aligned}$$

By Theorem B.1,

$$\sum_i u_i(v_i, b_{-i}) \geq e^{-\frac{2m}{L}} \sum_i u_i(\mathbf{x}_i(\mathbf{p}^*)).$$

Therefore,

$$\begin{aligned} \sum_i \sum_{t=1}^T u_i(b_i^t, b_{-i}^t) &\geq \sum_i \sum_{t=1}^T u_i(v_i, b_{-i}^t) - \sum_i \max_{i'} \Phi(|K_{i'}|, T) \cdot \lambda u_i(v_i, v_{-i}) \\ &\geq T \cdot e^{-\frac{2m}{L}} \sum_i u_i(\mathbf{x}_i(\mathbf{p}^*)) - \max_{i'} \Phi(|K_{i'}|, T) \lambda \sum_i u_i(v_i, v_{-i}). \end{aligned}$$

The theorem follows on dividing both sides by T . \square

C Reserve prices

Definition C.1. *Eisenberg Gale markets with reserve prices are exactly the solutions to the following convex program:*

$$\begin{aligned} \max_{\mathbf{x}} \quad & \sum_{i=1}^n e_i \cdot \log(u_i(x_{i1}, x_{i2}, \dots, x_{im})) + \sum_{j=1}^m y_j r_j \\ \text{s.t.} \quad & \forall j : \quad \sum_i x_{ij} + y_j \leq 1 \\ & \forall i, j : \quad x_{ij} \geq 0, \end{aligned}$$

where r_j is the reserve price of item j .

The proof of Theorem B.1 uses the following lemma.

Lemma C.1. *For any bidding profile \mathbf{b} and any value profile \mathbf{v} which are homogeneous of degree 1, concave, continuous, monotone and satisfy the gross substitutes property, if the reserve prices $r_j \leq \frac{1}{4}p_j^*$ for any j , then*

$$\sum_i e_i \log u_i(v_i, b_{-i}) - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}^*)) \geq -2m \cdot \max_{i'} e_{i'}.$$

Proof of Theorem B.1: On exponentiating the expressions on both sides in the statement of Lemma C.1 we obtain

$$\prod_i u_i(v_i, b_{-i})^{e_i} \geq \frac{1}{e^{2m \cdot \max_i e_i}} \prod_i u_i(v_i, v_{-i})^{e_i}.$$

Therefore,

$$\prod_i \left(\frac{u_i(v_i, b_{-i})}{u_i(v_i, v_{-i})} \right)^{e_i} \geq \frac{1}{e^{2m \cdot \max_i e_i}}.$$

Using the weighted GM-AM inequality, we obtain

$$\frac{\sum_i e_i \frac{u_i(v_i, b_{-i})}{u_i(v_i, v_{-i})}}{\sum_i e_i} \geq \left(\prod_i \left(\frac{u_i(v_i, b_{-i})}{u_i(v_i, v_{-i})} \right)^{e_i} \right)^{\frac{1}{\sum_i e_i}} \geq \left(\frac{1}{e^{2m \max_i e_i}} \right)^{\frac{1}{\sum_i e_i}} = e^{-\frac{2m \max_i e_i}{\sum_i e_i}}.$$

Since $u_i(v_i, v_{-i}) = te_i$, for all i ,

$$\sum_i u_i(v_i, b_{-i}) \geq e^{-\frac{2m \max_i e_i}{\sum_i e_i}} \sum_i u_i(v_i, v_{-i}).$$

□

Our goal is to bound $\sum_i e_i \log u_i(v_i, v_{-i}) - \sum_i e_i \log u_i(v_i, b_{-i})$. We will be working with the following function, the demand at prices \mathbf{p} :

$$x(\mathbf{p}) = (x_1(\mathbf{p}), x_2(\mathbf{p}), \dots) = \arg \max_{\mathbf{x}} \sum_i e_i \log u_i(\mathbf{x}_i) + 1 \cdot \mathbf{p} - \sum_i \mathbf{x}_i \cdot \mathbf{p}. \quad (\text{C.1})$$

Recall that, by Lemma 5.2, $p_j(v_i, b_{-i}) \leq p_j(b_i, b_{-i}) + 1 \cdot e_i$. Consequently, $u_i(\mathbf{x}_i(\mathbf{p}(v_i, b_{-i}))) \geq u_i(\mathbf{x}_i(\mathbf{p}(\mathbf{b}) + 1 \cdot \max_{i'} e_{i'}))$, and so it will suffice to bound $\sum_i e_i u_i(v_i, v_{-i}) - \sum_i e_i u_i(\mathbf{x}_i(\mathbf{p}(\mathbf{b}) + 1 \cdot \max_{i'} e_{i'}))$.

We want to apply the bound in (5.3), but then we need prices \mathbf{q} such that $\sum_j q_j = \sum_i e_i$. Accordingly, we will be considering the *scaled* prices $\mathbf{q}(\mathbf{b}) = (\mathbf{p}(\mathbf{b}) + 1 \cdot \max_{i'} e_{i'}) \cdot \frac{\sum_i \mathbf{x}_i}{\sum_j p_j(\mathbf{b}) + \max_{i'} e_{i'}}$ and the *compressed* prices, defined below.

For convenience, in the following definition, we set $\frac{0}{x} = 0$ for $x > 0$, $\frac{0}{0} = 1$ and $\frac{x}{0} = +\infty$ for $x > 0$.

Definition C.2. Let \mathbf{q} be a price vector such that $1 \cdot \mathbf{q} = \sum_i e_i$. The l -compressed version ($l \leq 1$) of \mathbf{q} is defined as $p'_j(l, \mathbf{q})$ where

$$\begin{aligned} \frac{p'_j(l, \mathbf{q})}{p_j^*} &= l & \text{if } \frac{q_j}{p_j^*} \leq l, \\ \frac{p'_j(l, \mathbf{q})}{p_j^*} &= t & \text{if } \frac{q_j}{p_j^*} \geq t, \\ \text{and } \frac{p'_j(l, \mathbf{q})}{p_j^*} &= \frac{q_j}{p_j^*} & \text{if } l < \frac{q_j}{p_j^*} < t, \end{aligned}$$

where t is a number bigger than 1 such that $\sum_j p'_j(l, \mathbf{q}) = \sum_i e_i$, and \mathbf{p}^* is the optimal solution ($1 \cdot \mathbf{p}^* = \sum_i e_i$).

Henceforth, unless noted otherwise, we let \mathbf{p} denote $\mathbf{p}(\mathbf{b})$ and \mathbf{q} denote $\mathbf{q}(\mathbf{b})$.

Lemma C.2.

$$\begin{aligned} \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p} + 1 \cdot \max_{i'} e_{i'})) &= \sum_i e_i \log u_i \left(\mathbf{x}_i(\mathbf{p} + 1 \cdot \max_{i'} e_{i'}) \cdot \frac{\sum_i e_i}{\sum_j (p_j + \max_{i'} e_{i'})} \right) \\ &\quad - \sum_i e_i \log \frac{\sum_j (p_j + \max_{i'} e_{i'})}{\sum_i e_i} \\ &= \sum_i e_i \log u_i(\mathbf{q}) - \sum_i e_i \log \frac{\sum_j (p_j + \max_{i'} e_{i'})}{\sum_i e_i}. \end{aligned}$$

Proof.

$$\begin{aligned}
\sum_i e_i \log u_i(x_i(\mathbf{p} + 1 \cdot \max_{i'} e_{i'})) &= \sum_i e_i \log \left[u_i(\mathbf{x}_i(\mathbf{p} + 1 \cdot \max_{i'} e_{i'})) \right. \\
&\quad \cdot \frac{\sum_j (p_j + \max_{i'} e_{i'})}{\sum_i e_i} \cdot \frac{\sum_i e_i}{\sum_j (p_j + \max_{i'} e_{i'})} \left. \right] \\
&= \sum_i e_i \log \left[u_i(\mathbf{x}_i((\mathbf{p} + 1 \cdot \max_{i'} e_{i'})) \cdot \frac{\sum_i e_i}{\sum_j p_j + \max_{i'} e_{i'}}) \cdot \frac{\sum_j e_i}{\sum_j p_j + \max_{i'} e_{i'}} \right].
\end{aligned}$$

Now

$$\begin{aligned}
&\sum_i e_i \log \left[u_i(\mathbf{x}_i((\mathbf{p} + 1 \cdot \max_{i'} e_{i'})) \cdot \frac{\sum_i e_i}{\sum_j p_j + \max_{i'} e_{i'}}) \cdot \frac{\sum_j e_i}{\sum_j p_j + \max_{i'} e_{i'}} \right] \\
&= \sum_i e_i \log u_i(\mathbf{x}_i((\mathbf{p} + 1 \cdot \max_{i'} e_{i'})) \cdot \frac{\sum_i e_i}{\sum_j p_j + \max_{i'} e_{i'}}) - \sum_i e_i \log \frac{\sum_j (p_j + \max_{i'} e_{i'})}{\sum_i e_i}.
\end{aligned}$$

□

Lemma C.3. Suppose that $\sum_j q_j = \sum_j p'_j = \sum_i e_i$. Then

$$\sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{q})) - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}')) \geq \sum_{ij} (p'_j - q_j) x_{ij}(\mathbf{p}').$$

Proof. As $\mathbf{x}(\mathbf{q}) = \arg \max_{\mathbf{x}} \sum_i e_i \log u_i(\mathbf{x}_i) - \sum_i \mathbf{x}_i \cdot \mathbf{q} + 1 \cdot \mathbf{q}$,

$$\begin{aligned}
&\sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{q})) - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}')) \\
&\geq \sum_i \mathbf{q} \cdot \mathbf{x}_i(\mathbf{q}) - \sum_i \mathbf{q} \cdot \mathbf{x}_i(\mathbf{p}').
\end{aligned}$$

As in the “PoA” analysis, $\mathbf{x}_i(\mathbf{q}) = \arg \max_{\mathbf{x}_i \cdot \mathbf{q} = e_i} u_i(\mathbf{x}_i)$. So, $\mathbf{x}_i(\mathbf{q}) \cdot \mathbf{q} = e_i$ and $\mathbf{x}_i(\mathbf{p}') \cdot \mathbf{p}' = e_i$. Therefore,

$$\begin{aligned}
&\sum_i \mathbf{q} \cdot \mathbf{x}_i(\mathbf{q}) - \sum_i \mathbf{q} \cdot \mathbf{x}_i(\mathbf{p}') \\
&= \sum_i \mathbf{q} \cdot \mathbf{x}_i(\mathbf{q}) - \sum_i \mathbf{q} \cdot \mathbf{x}_i(\mathbf{p}') + \sum_i \mathbf{p}' \cdot \mathbf{x}_i(\mathbf{p}') - \sum_i \mathbf{p}' \cdot \mathbf{x}_i(\mathbf{p}') \\
&= \sum_{ij} (p'_j - q_j) x_{ij}(\mathbf{p}').
\end{aligned}$$

□

Lemma C.4. There exists an $\mathbf{x}(\mathbf{p}')$, where $\mathbf{p}' = p'_j(l, \mathbf{q})$, such that, for any $l < 1$, if $\frac{p'_j(l, \mathbf{q})}{p_j^*} = l$, $\sum_i x_{ij}(\mathbf{p}') \geq \frac{1}{l}$ and if $\frac{p'_j(l, \mathbf{q})}{p_j^*} = t$, $\sum_i x_{ij}(\mathbf{p}') \leq \frac{1}{t}$.

Proof. It is straightforward to check this for linear utility functions.

Now we consider the single-demand WGS utility functions. Let $\hat{\mathbf{p}}$ denote the prices such that $\hat{p}_j = p_j$ when $p_j > 0$ and $\hat{p}_j = \epsilon$ when $p_j = 0$. Note that here ϵ is an arbitrarily small positive value.

By the homogeneity of the utility function, there exists an $\mathbf{x}(l\mathbf{p}^*)$, such that $\sum_i x_{ij}(l\mathbf{p}^*) = \frac{1}{l}$ for all j such that $p_j^* > 0$. Now, we consider $\sum_i x_{ij}(\widehat{l\mathbf{p}^*})$. For those j such that $p_j^* > 0$, by the Gross Substitutes property, $\sum_i x_{ij}(\widehat{l\mathbf{p}^*}) \geq \frac{1}{l}$.

Then, we let the price increase from $\widehat{l\mathbf{p}^*}$ to $\widehat{p'_j(l, \mathbf{q})}$. Also, by Gross Substitutes property, $\sum_i x_{ij}(\widehat{p'_j(l, \mathbf{q})}) \geq \frac{1}{l}$ for those j such that $p_j^* > 0$ and $\frac{p'_j(l, \mathbf{q})}{p_j^*} = l$. By the same reasoning, $\sum_i x_{ij}(\widehat{p'_j(l, \mathbf{q})}) \leq \frac{1}{t}$ for those j such that $p_j^* > 0$ and $\frac{p'_j(l, \mathbf{q})}{p_j^*} = t$.

Furthermore, by the Gross Substitutes property and homogeneity of the utility function, $\sum_i x_{ij}(\widehat{p'_j(l, \mathbf{q})}) = 0$ for those j such that $p_j^* = 0$.⁶

So, there exists an $\mathbf{x}(\hat{\mathbf{p}}')$, where $\mathbf{p}' = p'_j(l, \mathbf{q})$, such that for $p_j^* > 0$, if $\frac{p'_j(l, \mathbf{q})}{p_j^*} = l$, $\sum_i x_{ij}(\hat{\mathbf{p}}') \geq \frac{1}{l}$ and if $\frac{p'_j(l, \mathbf{q})}{p_j^*} = t$, $\sum_i x_{ij}(\hat{\mathbf{p}}') \leq \frac{1}{t}$, and for $p_j^* = 0$, $\sum_i x_{ij}(\hat{\mathbf{p}}') = 0$.

Since $l < 1$, $\frac{p'_j(l, \mathbf{q})}{p_j^*} \neq l$ when $p_j^* = 0$. Therefore, we have an $\mathbf{x}(\hat{\mathbf{p}}')$, where $\mathbf{p}' = p'_j(l, \mathbf{q})$, such that if $\frac{p'_j(l, \mathbf{q})}{p_j^*} = l$, $\sum_i x_{ij}(\hat{\mathbf{p}}') \geq \frac{1}{l}$ and if $\frac{p'_j(l, \mathbf{q})}{p_j^*} = t$, $\sum_i x_{ij}(\hat{\mathbf{p}}') \leq \frac{1}{t}$.

By the Gross Substitutes property, the demand $\mathbf{x}_i(\hat{\mathbf{p}}')$ for a given ϵ is also an optimal demand for any $0 < \epsilon' < \epsilon$. This is because for any small positive ϵ , the demand for those goods with price ϵ is 0. For reducing prices on those price ϵ goods only reduces the demand for other goods, but as there can be no reduction in spending on the latter goods, in fact the demands are unchanged.

Therefore, by the continuity and homogeneity of the utility function, there exists an optimal allocation $\mathbf{x}(\hat{\mathbf{p}}')$, which equals $\mathbf{x}(\hat{\mathbf{p}})$, where $\mathbf{p}' = p'_j(l, \mathbf{q})$. For if not, suppose there were a higher utility allocation when $\epsilon = 0$, whose value is u_0 , where $u_0 > u_i(\mathbf{x}_i(\hat{\mathbf{p}}'))$. Then at any $\epsilon > 0$, we could achieve utility $(1 - \lambda(\epsilon))u_0$, where $\lim_{\epsilon \rightarrow 0} \lambda(\epsilon) = 0$, and as $x_{ij}(\hat{\mathbf{p}}')$ is an optimal allocation, $u_i(\mathbf{x}_i(\hat{\mathbf{p}}')) \geq (1 - \lambda(\epsilon))u_0$. But this holds for every $\epsilon > 0$, so letting $\epsilon \rightarrow 0$ yields $u_i(\mathbf{x}_i(\hat{\mathbf{p}}')) \geq u_0$ and hence $\mathbf{x}(\hat{\mathbf{p}}')$ is an optimal allocation when $\epsilon = 0$. \square

Lemma C.5. Suppose that $\sum_j q_j = \sum_i e_i$. Then

$$\sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{q})) - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}'(l, \mathbf{q}))) \geq \sum_i \left(\frac{1}{l} - 1 \right) (l p_j^* - q_j) \cdot 1_{q_j \leq l p_j^*}.$$

Proof. By Lemma C.4, There exists an $\mathbf{x}(\hat{\mathbf{p}}')$, where $\mathbf{p}' = p'_j(l, \mathbf{q})$, such that if $\frac{p'_j(l, \mathbf{q})}{p_j^*} = l$,

⁶We consider a procedure that changes prices from p^* to $\widehat{p'_j(l, \mathbf{q})}$. First, we define prices $\mathbf{p}^{*''}$ such that $\frac{p'_j(l, \mathbf{q})}{p_j^{*''}} = k$ and $p_j^{*''} > 0$ if $p_j^* = 0$, and $\frac{p'_j(l, \mathbf{q})}{p_j^{*''}} \leq k$ and $p_j^{*''} = p_j^*$ if $p_j^* > 0$. Here, k is a positive constant and k is not infinity. We increase the prices from \mathbf{p}^* to $\mathbf{p}^{*''}$, by the Gross Substitutes property, $\sum_i x_{ij}(\mathbf{p}^{*''}) = 0$ for those j such that $p_j^* = 0$. Then, by homogeneity of the utility function, also for those j , $\sum_i x_{ij}(k\mathbf{p}^{*''}) = 0$. Now, we reduce the prices from $k\mathbf{p}^{*''}$ to $\widehat{p'_j(l, \mathbf{q})}$ ($k\mathbf{p}^{*''}$ is no less than $\widehat{p'_j(l, \mathbf{q})}$ by the definition of $\mathbf{p}^{*''}$). Since $\widehat{p'_j(l, \mathbf{q})} = k p_j^*$ for those j such that $p_j^* = 0$, by Gross Substitutes property, for those j , $\sum_i x_{ij}(\widehat{p'_j(l, \mathbf{q})}) \leq \sum_i x_{ij}(k\mathbf{p}^{*''})$. By previous argument that $\sum_i x_{ij}(k\mathbf{p}^{*''}) = 0$ for those j , $\sum_i x_{ij}(\widehat{p'_j(l, \mathbf{q})}) = 0$ also holds for those j .

$\sum_i x_{ij}(\mathbf{p}') \geq \frac{1}{l}$ and if $\frac{p'_j(l, \mathbf{q})}{p_j^*} = t$, $\sum_i x_{ij}(\mathbf{p}') \leq \frac{1}{t}$. Therefore, by lemma C.3,

$$\begin{aligned} & \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{q})) - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}'(l, \mathbf{q}))) \\ & \geq \sum_{ij} (p'_j(l, \mathbf{q}) - q_j) x_{ij}(\mathbf{p}'(l, \mathbf{q})) \\ & \geq \sum_j (p'_j(l, \mathbf{q}) - q_j) \cdot \mathbf{1}_{\frac{q_j}{p_j^*} \leq l} \cdot \frac{1}{l} + \sum_j (p'_j(l, \mathbf{q}) - q_j) \cdot \mathbf{1}_{\frac{q_j}{p_j^*} \geq t} \cdot \frac{1}{t}. \end{aligned}$$

Since $\sum_j p'_j(l, \mathbf{q}) = \sum_i e_i = \sum_j q_j$, and as $q_j = p'_j(l, \mathbf{q})$ when $l < \frac{q_j}{p_j^*} < t$,

$$\sum_j (p'_j(l, \mathbf{q}) - q_j) \cdot \mathbf{1}_{\frac{q_j}{p_j^*} \leq l} = - \sum_j (p'_j(l, \mathbf{q}) - q_j) \cdot \mathbf{1}_{\frac{q_j}{p_j^*} \geq t}.$$

Thus

$$\begin{aligned} \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{q})) - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}'(l, \mathbf{q}))) & \geq \sum_j \left(\frac{1}{l} - \frac{1}{t} \right) (p'_j(l, \mathbf{q}) - q_j) \cdot \mathbf{1}_{\frac{q_j}{p_j^*} \leq l} \\ & \geq \sum_j \left(\frac{1}{l} - 1 \right) (l p_j^* - q_j) \cdot \mathbf{1}_{q_j \leq l p_j^*}. \end{aligned}$$

□

Lemma C.6. $\sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}'(l, \mathbf{q}))) \geq \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}^*))$.

Proof. The result follows from (5.3), as $\sum_j p_j^* = \sum_i e_i = \sum_j p'_j$. □

The proof of the next Corollary uses Lemma 5.2, which depends on Lemma 5.3, whose proof, for single-demand utility functions, changes slightly in the presence of reserve prices, as we comment on next.

REVISED PROOF OF LEMMA 5.3. There are two changes. First, when $l \in S$, we can conclude that $p_l = p'_l > \max\{0, r_l\}$ (in the line preceding (5.4)). Second, when $p_l > \max\{0, r_l\}$, $\sum_{h \neq i} x'_{hl} = \sum_{h \neq i} x_{hl} = 1$ (in the line preceding (5.5)). These are the only changes.

Corollary C.1.

$$\begin{aligned} & \sum_i e_i \log u_i(v_i, b_{-i}) - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}^*)) \\ & \geq \sum_j (1 - l) p_j^* - \left(\frac{1}{l} - 1 \right) (p_j + \max_{i'} e_{i'}) \frac{\sum_i e_i}{\sum_j (p_j + \max_{i'} e_{i'})} \cdot \mathbf{1}_{(p_j + \max_{i'} e_{i'}) \frac{\sum_i e_i}{\sum_j (p_j + \max_{i'} e_{i'})} \leq l p_j^*} \\ & \quad - \sum_i e_i \frac{\sum_j (p_j + \max_{i'} e_{i'})}{\sum_i e_i}. \end{aligned}$$

Proof.

$$\begin{aligned}
& \sum_i e_i \log u_i(v_i, b_{-i}) - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}^*)) \\
& \geq \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p} + 1 \cdot \max_{i'} e_{i'})) - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}^*)) \\
& \geq \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{q})) - \sum_i e_i \log \frac{\sum_j (p_j + \max_{i'} e_{i'})}{\sum_i e_i} \\
& \quad - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}^*)) \quad (\text{by Lemma C.2}) \\
& \geq \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}'(l, \mathbf{q}))) + \sum_i \left(\frac{1}{l} - 1 \right) (lp_j^* - q_j) \cdot \mathbf{1}_{q_j \leq lp_j^*} \\
& \quad - \sum_i e_i \log \frac{\sum_j (p_j + \max_{i'} e_{i'})}{\sum_i e_i} - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}^*)) \quad (\text{by Lemma C.5}) \\
& \geq \sum_i \left(\frac{1}{l} - 1 \right) (lp_j^* - q_j) \cdot \mathbf{1}_{q_j \leq lp_j^*} - \sum_i e_i \log \frac{\sum_j (p_j + \max_{i'} e_{i'})}{\sum_i e_i} \quad (\text{by Lemma C.6}).
\end{aligned}$$

□

Let $\delta = \sum_j (p_j + \max_{i'} e_{i'}) - \sum_i e_i$. Suppose that the good j reserve price $r_j \leq \frac{1}{4}p_j^*$ for all j . Note that $\sum_i e_i + \sum_j r_j \cdot \mathbf{1}_{p_j = r_j} \geq \sum_j p_j \geq \sum_i e_i$. Clearly, $\sum_j r_j \cdot \mathbf{1}_{p_j = r_j} \geq \delta - m \cdot \max_{i'} e_{i'}$.

Lemma C.7.

$$\sum_i e_i \log \frac{\sum_j (p_j + \max_{i'} e_{i'})}{\sum_i e_i} \leq \delta.$$

Proof.

$$\sum_i e_i \log \frac{\sum_j (p_j + \max_{i'} e_{i'})}{\sum_i e_i} = \sum_i e_i \log \left(1 + \frac{\delta}{\sum_i e_i} \right) \leq \sum_i e_i \frac{\delta}{\sum_i e_i} = \delta.$$

□

Proof of Lemma C.1: We set $l = \frac{1}{2}$. Then by Corollary C.1 and Lemma C.7,

$$\begin{aligned}
& \sum_i e_i \log u_i(v_i, b_{-i}) - \sum_i e_i \log u_i(\mathbf{x}_i(\mathbf{p}^*)) \\
& \geq \sum_j (2-1) \left(\frac{1}{2} p_j^* - (p_j + \max_{i'} e_{i'}) \cdot \frac{\sum_i e_i}{\sum_j (p_j + \max_{i'} e_{i'})} \right) \\
& \quad \cdot 1_{(p_j + \max_{i'} e_{i'}) \cdot \frac{\sum_i e_i}{\sum_j (p_j + \max_{i'} e_{i'})} \geq \frac{1}{2} p_j^* - \delta} \\
& \geq \sum_j (2-1) \left(\frac{1}{2} p_j^* - (r_j + \max_{i'} e_{i'}) \cdot \frac{\sum_i e_i}{\sum_j (p_j + \max_{i'} e_{i'})} \right) \\
& \quad \cdot 1_{p_j = r_j \wedge (r_j + \max_{i'} e_{i'}) \cdot \frac{\sum_i e_i}{\sum_j (p_j + \max_{i'} e_{i'})} \leq \frac{1}{2} p_j^* - \delta} \\
& \geq \sum_j \left(\frac{1}{2} p_j^* - (r_j + \max_{i'} e_{i'}) \cdot \frac{\sum_i e_i}{\sum_j (p_j + \max_{i'} e_{i'})} \right) \cdot 1_{p_j = r_j \wedge (r_j + \max_{i'} e_{i'}) \leq \frac{1}{2} p_j^* - \delta} \\
& \quad (\text{as } \frac{\sum_i e_i}{\sum_j (p_j + \max_{i'} e_{i'})} \leq 1) \\
& \geq \sum_j \left(\frac{1}{2} p_j^* - (r_j + \max_{i'} e_{i'}) \right) \cdot 1_{p_j = r_j \wedge (r_j + \max_{i'} e_{i'}) \leq \frac{1}{2} p_j^* - \delta} \quad (\text{as } \frac{\sum_i e_i}{\sum_j (p_j + \max_{i'} e_{i'})} \leq 1) \\
& \geq \sum_j \left(\frac{1}{2} p_j^* - (r_j + \max_{i'} e_{i'}) \right) \cdot 1_{p_j = r_j \wedge r_j \leq \frac{1}{2} p_j^* - \delta} \\
& \geq \sum_j r_j \cdot 1_{p_j = r_j} - m \cdot \max_{i'} e_{i'} - \delta \quad (\text{as } \frac{1}{2} p_j^* \geq 2r_j) \\
& \geq \delta - m \cdot \max_{i'} e_{i'} - m \cdot \max_{i'} e_{i'} - \delta = -2m \cdot \max_{i'} e_{i'} \quad (\text{as } \sum_j r_j \cdot 1_{p_j = r_j} \geq \delta - m \cdot \max_{i'} e_{i'}).
\end{aligned}$$

□